On the universality of critical exponents

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Abstract

After reviewing the essential structure of ERG differential equations, we show that the critical exponents defined at a fixed point of RG flows are independent of the choice of cutoff functions.

Introduction

1. **Universality** is a key concept not only for critical phenomena but also for construction of **continuum limits** in quantum field theory.

2. A universality class is determined by a fixed point of RG flows, which is characterized by **critical exponents**.

3. Universality has been derived within the context of ERG. (Morris, Lattore, Rosten, ...)

4. We review the basic structure of ERG dependent on two arbitrary cutoff functions, one for wave function renormalization and another for partial integration of fluctuations.

5. We consider only a real scalar field. Generalization that includes more fields (bosonic or fermionic) is straightforward.

Let's get back to the basics of ERG.

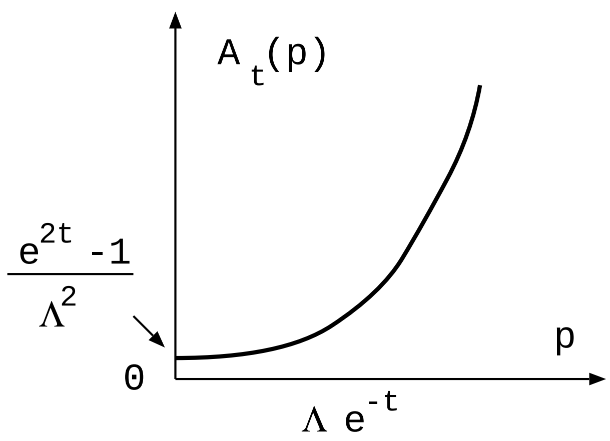
$S[\phi]$ is a Wilson action.

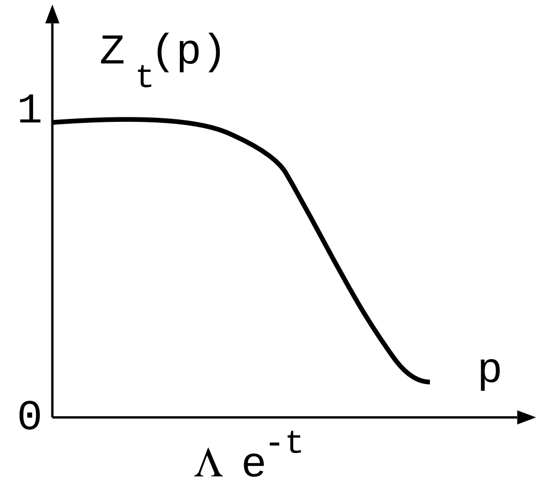
$$\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle_S \equiv \frac{\int [d\phi]\, \phi(p_1) \cdots \phi(p_n)\, \mathrm{e}^{S[\phi]}}{\int [d\phi]\, \mathrm{e}^{S[\phi]}}$$

Generalized diffusion equations

1. Diffusion in the space of field configurations

$$\mathrm{e}^{S_t[\phi]} = \int [d\phi'] \exp\left[-\frac{1}{2} \int_p \frac{1}{A_t(p)} \left(\frac{\phi(p)}{\sqrt{Z_t(p)}} - \phi'(p) \right) \left(\frac{\phi(-p)}{\sqrt{Z_t(p)}} - \phi'(-p) \right) \right] \mathrm{e}^{S[\phi']}$$





2. **ERG differential equation**

$$\partial_t S_t[\phi] = \int_p \left(-\frac{1}{2}\frac{\partial_t Z_t(p)}{Z_t(p)}\phi(p)\frac{\delta}{\delta\phi(p)} + Z_t(p)\partial_t A_t(p)\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \mathrm{e}^{S_t[\phi]}$$

3. We introduce **two cutoff functions** K & k

$$\sqrt{Z_t(p)} = \frac{K\left(\frac{p}{\Lambda \mathrm{e}^{-t}}\right)}{K\left(\frac{p}{\Lambda}\right)}, \quad A_t(p) = \frac{1}{p^2}\left(k\left(\frac{p}{\Lambda\mathrm{e}^{-t}}\right)\frac{1}{Z_t(p)} - k\left(\frac{p}{\Lambda}\right)\right)$$

so that

$$\begin{cases} -\frac{1}{2}\partial_t \ln Z_t(p) = \dfrac{\Delta\left(\frac{p}{\Lambda\mathrm{e}^{-t}}\right)}{K\left(\frac{p}{\Lambda\mathrm{e}^{-t}}\right)}, \quad \text{where } \Delta(p) \equiv -2p^2 \frac{d}{dp^2}K(p) \\ Z_t(p)\partial_t A_t(p) = \frac{1}{p^2}\, g\left(\frac{p}{\Lambda\mathrm{e}^{-t}}\right), \quad \text{where } g(p) \equiv 2\frac{\Delta(p)}{K(p)}\, k(p) + 2p^2\frac{d}{dp^2}k(p) \end{cases}$$

$$\text{Examples:} \quad k(p) = p^2\,(\text{Wilson}), \quad K(p)\,(1-K(p))\ (\text{Polchinski})$$

4. Generating functional

$$\mathrm{e}^{W_t[J]} \equiv \int [d\phi] \exp\left(S_t[\phi] + \int_p J(-p)\phi(p) \right)$$

satisfies

$$W_t[J] = W_0[\sqrt{Z_t}\, J] + \frac{1}{2}\int_p A_t(p) Z_t(p) J(-p) J(p)$$

5. t-independent correlation functions

$$\begin{cases} \langle\langle \phi(p)\phi(q) \rangle\rangle & \equiv \frac{1}{K\left(\frac{p}{\Lambda \mathrm{e}^{-t}}\right)^2} \left\{ \langle\, \phi(p)\phi(q)\, \rangle_{S_t} - \frac{k\left(\frac{p}{\Lambda \mathrm{e}^{-t}}\right)}{p^2} \delta(p+q) \right\} \\ \langle\langle \phi(p_1)\cdots\phi(p_n) \rangle\rangle^c & \equiv \prod_{i=1}^n \frac{1}{K\left(\frac{p_i}{\Lambda \mathrm{e}^{-t}}\right)} \cdot \langle\, \phi(p_1)\cdots\phi(p_n)\, \rangle_{S_t}^c \end{cases}$$

i.e., S_t is equivalent with the original S.

ERG for fixed points

1. Modify the ERG:

   (a) choose a positive constant $\mathcal{Z}_t$ to normalize the kinetic term of S_t

$$\begin{cases} \sqrt{Z_t(p)} &= \frac{K\left(\frac{p}{\Lambda e^{-t}}\right)}{K\left(\frac{p}{\Lambda}\right)} \sqrt{\mathcal{Z}_t} \quad \text{where } \mathcal{Z}_t = \exp\left(2 \int^t ds\, \gamma_s\right) \\ A_t(p) &= \frac{1}{p^2}\left(k\left(\frac{p}{\Lambda e^{-t}}\right)\frac{1}{Z_t(p)} - k\left(\frac{p}{\Lambda}\right)\right) \end{cases}$$

   (b) make it dimensionless — substitute $\left(\Lambda e^{-t}\right)^{-\frac{D+2}{2}} \phi\left(\frac{p}{\Lambda e^{-t}}\right)$ into $\phi(p)$

2. ERG diff eq for fixed points

$$\partial_t e^{S_t[\phi]} = \int_p \Big[\left\{ p_\mu \frac{\partial \phi(p)}{\partial p_\mu} + \left(\frac{D+2}{2} + \frac{\Delta(p)}{K(p)} - \gamma_t \right) \phi(p) \right\} \frac{\delta}{\delta \phi(p)} + \frac{1}{p^2}\left(g(p) - 2\gamma_t\, k(p)\right) \frac{1}{2} \frac{\delta^2}{\delta\phi(-p)\delta\phi(p)} \Big] e^{S_t[\phi]}$$

3. **The anomalous dimension γ_t is determined by S_t.** For example, for Wilson's choice $k(p) = p^2, K(p) = \mathrm{e}^{-p^2}$, we find

$$2\gamma_t = \frac{4\mathcal{V}_2(0)^2 + \frac{1}{2}\frac{\partial}{\partial p^2}\int_q (2+4q^2)\mathcal{V}_4(q,-q,p,-p)\Big|_{p=0}}{-1 - 2\mathcal{V}_2(0) + \frac{1}{2}\frac{\partial}{\partial p^2}\int_q \mathcal{V}_4(q,-q,p,-p)\Big|_{p=0}}$$

4. **Scaling laws**: the correlation functions

$$\left\{ \begin{array}{ccc} \langle\!\langle \phi(p)\phi(q) \rangle\!\rangle_t & \equiv & \frac{1}{K(p)^2}\left\{ \langle\, \phi(p)\phi(q)\,\rangle_{S_t} - \frac{k(p)}{p^2}\delta(p-q)\right\} \\ \langle\!\langle \phi(p_1)\cdots\phi(p_n) \rangle\!\rangle_t & \equiv & \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S_t} \end{array} \right.$$

satisfy

$$\left\langle\!\!\left\langle \phi(p_1\mathrm{e}^{\Delta t})\cdots\phi(p_n\mathrm{e}^{\Delta t}) \right\rangle\!\!\right\rangle_{t+\Delta t}$$
$$= \mathrm{e}^{-n\frac{D+2}{2}\Delta t}\exp\left(n\int_t^{t+\Delta t} ds\,\gamma_s \right) \langle\!\langle \phi(p_1)\cdots\phi(p_n) \rangle\!\rangle_t$$

Fixed points

1. For $2 \leq D < 4$, both Gauss and Wilson-Fisher fixed points exist.

2. A fixed point action S^* satisfies

$$\int_p \Bigg[\left\{ p_\mu \frac{\partial \phi(p)}{\partial p_\mu} + \left(\frac{D+2}{2} + \frac{\Delta(p)}{K(p)} - \gamma^* \right) \phi(p) \right\} \frac{\delta}{\delta \phi(p)} + \frac{1}{p^2} \left(g(p) - 2\gamma^* \, k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(-p) \delta \phi(p)} \Bigg] \mathrm{e}^{S^*[\phi]} = 0$$

which is characterized by γ^*.

3. **Scaling laws**

$$\left\langle\!\!\left\langle \phi(p_1 \mathrm{e}^t) \cdots \phi(p_n \mathrm{e}^t) \right\rangle\!\!\right\rangle_{S^*} = \exp \left(n \left(- \frac{D+2}{2} + \gamma^* \right) t \right) \cdot \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S^*}$$

Composite operators at a fixed point

1. Composite operators are infinitesimal variations of the Wilson action.

$$\partial_t \left(\mathcal{O}_t[\phi] \mathrm{e}^{S_t[\phi]} \right) = \int_p \Big[\left\{ p_\mu \frac{\partial \phi(p)}{\partial p_\mu} + \left(\frac{D+2}{2} + \frac{\Delta(p)}{K(p)} - \gamma_t \right) \phi(p) \right\} \frac{\delta}{\delta \phi(p)} + \frac{1}{p^2} \left(g(p) - 2\gamma_t\, k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta\phi(-p)\delta\phi(p)} \Big] \left(\mathcal{O}_t[\phi] \mathrm{e}^{S_t[\phi]} \right)$$

2. The correlation functions

$$\langle\!\langle \mathcal{O}_t\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_t^c \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \langle\, \mathcal{O}_t[\phi]\, \phi(p_1) \cdots \phi(p_n)\, \rangle_{S_t}^c$$

satisfy

$$\left\langle\!\!\left\langle \mathcal{O}_{t+\Delta t}\, \phi(p_1 \mathrm{e}^{\Delta t}) \cdots \phi(p_n \mathrm{e}^{\Delta t}) \right\rangle\!\!\right\rangle_{t+\Delta t}^c = \mathrm{e}^{n\left(-\frac{D+2}{2} + \int_t^{t+\Delta t} ds\, \gamma_s \right)} \langle\!\langle \mathcal{O}_t\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_t^c$$

3. **Number counting operator** (redundant operator, eq of motion op)

$$\mathcal{N}_t[\phi]\mathrm{e}^{S_t} \equiv -\int_p \left(\phi(p)\frac{\delta}{\delta\phi(p)} + \frac{k(p)}{p^2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \mathrm{e}^{S_t}$$

satisfies

$$\langle\!\langle \mathcal{N}_t\, \phi(p_1)\cdots\phi(p_n) \rangle\!\rangle_t = n\, \langle\!\langle \phi(p_1)\cdots\phi(p_n) \rangle\!\rangle_t$$

4. At a fixed point, $\partial_t S^* = 0$ implies

$$\begin{aligned} \partial_t \mathcal{O}_t[\phi]\cdot \mathrm{e}^{S^*} &= \int_p \Big[\Big\{ p_\mu \frac{\partial\phi(p)}{\partial p_\mu} + \left(\frac{D+2}{2} + \frac{\Delta(p)}{K(p)} - \gamma^* \right)\phi(p) \Big\} \frac{\delta}{\delta\phi(p)} \\ &\quad + \frac{1}{p^2}\left(g(p) - 2\gamma^*\, k(p) \right) \frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \Big] \left(\mathcal{O}_t[\phi]\mathrm{e}^{S^*} \right) \\ &\equiv \left(\mathcal{D}\mathcal{O}_t[\phi] \right) \mathrm{e}^{S^*} \end{aligned}$$

5. Eigenvalue equations

$$\mathcal{D}\mathcal{O}^y[\phi] = y\,\mathcal{O}^y[\phi]$$

imply the scaling laws

$$\left\langle\!\!\left\langle \mathcal{O}^y\,\phi(p_1 \mathrm{e}^t)\cdots\phi(p_n \mathrm{e}^t)\right\rangle\!\!\right\rangle_{S^*} = \mathrm{e}^{t\left(-y+n\left(-\frac{D+2}{2}+\gamma^*\right)\right)}\,\langle\!\langle \mathcal{O}^y\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S^*}$$

6. Ambiguity of $\mathcal{O}^y$

$$\langle\!\langle \left(\mathcal{N}^* * \mathcal{O}^y\right)\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S^*} = n\,\langle\!\langle \mathcal{O}^y\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S^*}$$

$\mathcal{O}^y + c\,\mathcal{N}^* * \mathcal{O}^y$ satisfies the same eigenvalue equation as $\mathcal{O}^y$.

7. $\mathcal{O}^y$ is determined uniquely (up to a normalization) if the absence of a kinetic term is required.

Linearlization of ERG near S^*

1. Consider a small variation ($|g_i(t)| \ll 1$)

$$S_t = S^* + \sum_i g_i(t) \mathcal{O}_i \quad \text{where } \mathcal{D}\mathcal{O}_i = y_i \mathcal{O}_i$$

   where $\mathcal{O}_i$ has no kinetic term.

2. The ERG diff eq with $\gamma_t = \gamma^*$ gives rise to

$$\frac{d}{dt} g_i(t) = y_i \, g_i(t)$$

3. g_i is **relevant** if $y_i > 0$, **irrelevant** if $y_i < 0$.

Proof of universality

1. Universality amounts to the independence of the critical exponents γ^* & y from the choice of cutoff functions K and k.

2. Infinitesimal changes δK & δk can be compensated by

$$\begin{aligned}\delta S_t &= \int_p \Big[-\frac{\delta K(p)}{K(p)}\phi(p)\frac{\delta S}{\delta\phi(p)} \\ &\quad + \frac{1}{p^2}\left(\delta k(p) - 2k(p)\frac{\delta K(p)}{K(p)}\right)\frac{1}{2}\left\{\frac{\delta S_t}{\delta\phi(-p)}\frac{\delta S_t}{\delta\phi(p)} + \frac{\delta^2 S_t}{\delta\phi(-p)\delta\phi(p)}\right\}\Big]\end{aligned}$$

3. Choose ϵ_t so that $\delta' S_t \equiv \delta S_t + \epsilon_t \mathcal{N}_t$ has no kinetic term.

4. Equivalence up to normalization

$$\langle\langle \phi(p_1)\cdots\phi(p_n)\rangle\rangle^{K+\delta K, k+\delta k}_{S_t+\delta' S_t} = (1 + n\epsilon_t)\,\langle\langle \phi(p_1)\cdots\phi(p_n)\rangle\rangle^{K,k}_{S_t}$$

5. $S_t + \delta' S_t$ satify the ERG diff eq with the anomalous dimension $\gamma_t + \delta\gamma_t$ where

$$\delta\gamma_t = \frac{d}{dt}\epsilon_t$$

6. This implies $\delta\gamma^* = 0 \implies$ **universality of** γ^*

7. For an arbitrary composite operator $\mathcal{O}_t$, we choose

$$\delta\mathcal{O}_t = -\int_p \frac{\delta K(p)}{K(p)} \frac{\delta}{\delta\phi(p)} \left(\phi(p)\mathcal{O}_t \mathrm{e}^{S_t}\right) \cdot \mathrm{e}^{-S_t} - \mathcal{O}_t \delta' S_t$$

so that

$$\langle\!\langle (\mathcal{O}_t + \delta\mathcal{O}_t)\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S_t+\delta' S_t}^{K+\delta K, k+\delta k} = \langle\!\langle \mathcal{O}_t\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S_t}^{K,k}$$

8. We choose ϵ^y so that

$$\delta' \mathcal{O}^y \equiv \delta \mathcal{O}^y + \epsilon^y \mathcal{N} * \mathcal{O}^y$$

has no kinetic term.

9. At the fixed point $S^* + \delta' S^*$, $\mathcal{O}^y + \delta' \mathcal{O}^y$ obeys the same scaling laws:

$$\left\langle\!\!\left\langle (\mathcal{O}^y + \delta' \mathcal{O}^y) \phi(p_1 \mathrm{e}^t) \cdots \phi(p_n \mathrm{e}^t) \right\rangle\!\!\right\rangle_{S^*+\delta' S^*}^{K+\delta K, k+\delta k}$$
$$= \mathrm{e}^{t\left(-y+n\left(-\frac{D+2}{2}+\gamma^*\right)\right)} \left\langle\!\left\langle (\mathcal{O}^y + \delta' \mathcal{O}^y) \phi(p_1) \cdots \phi(p_n) \right\rangle\!\right\rangle_{S^*+\delta' S^*}^{K+\delta K, k+\delta k}$$

10. Thus, y is independent of K & k. $\Longrightarrow$ **universality of** y

Conclusions

1. We have considered ERG differential equations with two cutoff functions K & k (or A & Z).

2. We have shown the independence of critical exponents $\eta = 2\gamma^*$ & y from the choices of K & k.

Appendix: Special composite operators

1. $[\mathcal{O}\phi(p)]$

$$[\mathcal{O}\phi(p)]\,\mathrm{e}^S \equiv \mathcal{O}\frac{1}{K(p)}\left(\phi(p) + \frac{k(p)}{p^2}\frac{\delta}{\delta\phi(-p)}\right)\mathrm{e}^S$$

satisfies

$$\langle\!\langle [\mathcal{O}\phi(p)]\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle = \langle\!\langle \mathcal{O}\,\phi(p)\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle$$

2. $\mathcal{N} * \mathcal{O}$

$$\begin{aligned}(\mathcal{N} * \mathcal{O})\,\mathrm{e}^S &\equiv -\int_p K(p)\frac{\delta}{\delta\phi(p)}\left([\mathcal{O}\phi(p)]\,\mathrm{e}^S\right)\\ &= -\int_p \frac{\delta}{\delta\phi(p)}\left\{\left(\phi(p) + \frac{k(p)}{p^2}\frac{\delta}{\delta\phi(-p)}\right)\left(\mathcal{O}\,\mathrm{e}^S\right)\right\}\end{aligned}$$

satisfies

$$\langle\!\langle \mathcal{N} * \mathcal{O}\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle = n\,\langle\!\langle \mathcal{O}\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle$$

3. Scaling laws in terms of composite operators

$$-\mathrm{e}^{-S^*}\int_p K(p)\frac{\delta}{\delta\phi(p)}\left(p_\mu\frac{\partial}{\partial p_\mu}[\phi](p)\,\mathrm{e}^{S^*}\right) = \left(\gamma^* - \frac{D+2}{2}\right)\mathcal{N}^*$$

4. conformal invariance?