The Compactly Supported Smooth Regulator

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ERG Lefkada 2014

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Functional Renormalization Group

- Wetterich RG equation

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{k\partial_k R_k}{\Gamma_k^{(2)} + R_k}\right)$$

- Gradient (derivative) expansion

$$\Gamma_k[\varphi] = \int_x \left[V_k(\varphi) + Z_k(\varphi)\frac{1}{2}(\partial_\mu \varphi)^2 + \ldots\right].$$

- Regulator function $\quad (R_k(p) \equiv p^2\, r(y),\ \ y = \frac{p^2}{k^2})$

$$\begin{aligned} R_{k\to 0}(p) &= 0, \\ R_{k\to \Lambda}(p) &= \infty, \\ R_k(p\to 0) &> 0 \end{aligned}$$

- Power-law regulator

$$r_{\mathrm{pow}}(y) = \frac{a}{y^b}$$

$\Longrightarrow$ compatible with the derivative expansion,
$\Longrightarrow$ depends on the upper bound of momentum integration

- Exponential regulator

$$r_{\mathrm{exp}}(y) = \frac{a}{\exp\left[c\ y^b\right] - 1}$$

$\Longrightarrow$ compatible with the derivative expansion,
$\Longrightarrow$ depends on the upper bound of momentum integration

- Litim's regulator

$$r_{\mathrm{opt}}(y) = a\left(\frac{1}{y^b} - 1\right)\Theta(1-y)$$

$\Longrightarrow$ momentum integral can be performed analytically,
$\Longrightarrow$ confront to the derivative expansion,

Compactly Supported Smooth (CSS) regulator

- general CSS regulator

$$r_{\rm css}^{\rm gen}(y) = \frac{\exp[cy_0^b/(f-hy_0^b)]-1}{\exp[cy^b/(f-hy^b)]-1}\theta(f-hy^b)$$

I. N., JHEP **04** (2013) 150

- normalised CSS regulator ($f \equiv 1$ and y_0 fixed)

$$r_{\rm css}^{\rm norm}(y) = \frac{\exp[\ln(2)c]-1}{\exp\left[\frac{\ln(2)cy^b}{1-hy^b}\right]-1}\theta(1-hy^b) = \frac{2^c-1}{2^{\frac{c\,y^b}{1-hy^b}}-1}\theta(1-hy^b)$$

I. N., I. G. Márián, V. Bacsó, PRD **89** (2014) 047701

$\Longrightarrow$ smooth: compatible with the derivative expansion,
$\Longrightarrow$ compact support: momentum integral is bounded,
$\Longrightarrow$ momentum integral cannot be performed analytically

- Properties of the general CSS regulator

$$\lim_{c\to 0} r_{\mathrm{css}}^{\mathrm{gen}} = = \frac{y_0^b}{1-y_0^b}\left(\frac{1}{y^b}-1\right)\Theta(1-y),$$

$$\lim_{f\to\infty} r_{\mathrm{css}}^{\mathrm{gen}} = = \frac{y_0^b}{y^b},$$

$$\lim_{h\to 0, c=f} r_{\mathrm{css}}^{\mathrm{gen}}(y) = = \frac{\exp[y_0^b]-1}{\exp[y^b]-1}.$$

- Properties of the normalised CSS regulator

$$\lim_{c\to 0, h\to 1} r_{\mathrm{css}}^{\mathrm{norm}} = \left(\frac{1}{y^b}-1\right)\theta(1-y)$$

$$\lim_{c\to 0, h\to 0} r_{\mathrm{css}}^{\mathrm{norm}} = \frac{1}{y^b}$$

$$\lim_{c\to 1, h\to 0} r_{\mathrm{css}}^{\mathrm{norm}} = \frac{1}{\exp[\ln(2)y^b]-1}$$

Optimization I.

- Amplitude ($\omega = k^{-2} V_k''$) expansion in LPA

$$k\partial_k V_k = -\alpha_d k^d \int_0^\infty \mathrm{d}y \frac{r' \, y^{1+\frac{d}{2}}}{(1+r)y+\omega} = \sum_{m=1}^{\infty} \frac{2m}{d} \; a_{2m-d} \; (-\omega)^{m-1}$$

$\Longrightarrow$ best convergence: Litim's regulator (b=1, a=1)
$\Longrightarrow$ confront to the derivative expansion

- Shortest Trajectory in the Theory Space

Jan M. Pawlowski, Annals of Physics **322** (2007) 2831

$\Longrightarrow$ works in any order of the derivative expansion,
$\Longrightarrow$ it gives the Litim regulator in LPA,
$\Longrightarrow$ no explicit r(y) beyond LPA

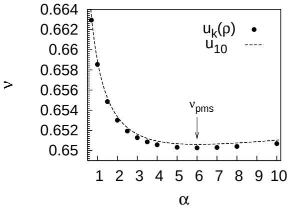
- CSS is differentiable, recovers the Litim regulator in LPA.

Optimization II.

- Principle of Minimal Sensitivity (PMS):

L. Canet, B. Delamotte, D. Mouhanna and J. Vidal, PRD **67** (2003) 065004

$$r(y) = \alpha \frac{1}{e^y - 1}$$
$$\Longrightarrow \alpha_{opt} = 6$$



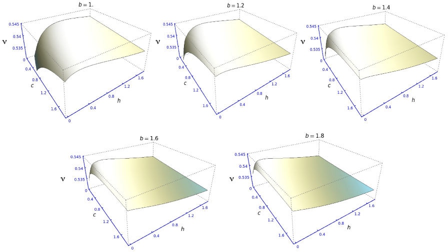
Léonie Canet, PRB **71** (2005) 012418

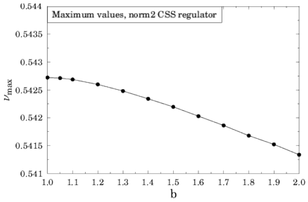
- optimal choice for the parameters for a given regulator
- properties:
  $\Longrightarrow$ works in any order of the derivative expansion,
  $\Longrightarrow$ regulators cannot be compared directly
- CSS provides a framework to compare regulators directly.

Model: O(1), d=3, LPA

$$\Gamma_k[\varphi] = \int d^3x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \sum_{n=1}^{N_{\text{cut}}} \frac{g_n(k)}{n!} \varphi^n \right]$$

Critical exponent ν for the 3D O(1) model (regulator in exponential normalization, b parameter fixed)



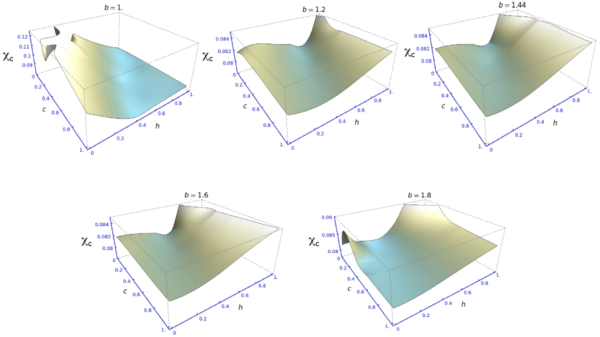


Best: $c = 0.001$, $h = 1$, $b = 1 \Longrightarrow$ Litim limit

Model: QED$_2$, d=2, LPA

$$\Gamma_k[\varphi] = \int d^2x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} M_k^2 \varphi^2 + u_k \cos(\sqrt{4\pi}\varphi) \right]$$

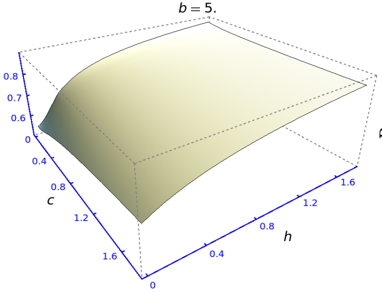
Critical ratio χ_c for bosonized QED$_2$ (regulator in exponential normalization, b parameter fixed)

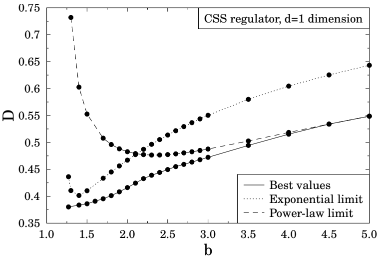


Best: $c = 0.001$, $h = 1$, $b = 1 \implies$ Litim limit

Model: sine-Gordon, d=1, LPA + z

$$\Gamma_k[\varphi] = \int dx \left[\frac{1}{2} z_k (\partial_\mu \varphi)^2 + u_k \cos(\varphi)\right]$$





Best: $c \to 0$ (small but non-zero), $b \to 1 \implies$ Litim limit

Summary

- Wetterich RG equation

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\,\frac{k\partial_k R_k}{\Gamma_k^{(2)} + R_k}, \qquad R_k(p) \equiv p^2\, r(y), \;\; y = \frac{p^2}{k^2}$$

- CSS regulator – "unification" of regulator functions

$$r_{\mathrm{css}}^{\mathrm{norm}}(y) = \frac{\exp[\ln(2)c] - 1}{\exp\left[\frac{\ln(2)cy^b}{1-hy^b}\right] - 1}\theta(1-hy^b) = \frac{2^c - 1}{2^{\frac{c\,y^b}{1-hy^b}} - 1}\theta(1-hy^b)$$

- single numerical code for all regulators
- no problem with the upper bound of the momentum integral
- approximation to the Litim-Pawlowski scheme beyond LPA
- regulators can be compared through the PMS

Outlook

optimization of CSS (best: Litim limit?)