Functional renormalization group and statistical mechanics of membranes

O. Zanusso

TPI Jena, Germany

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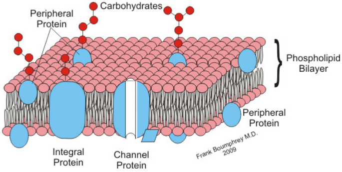
Outline

- Mechanical Membranes
  - Mechanical phases of membranes
  - Order parameters from the microscopic description
  - Effective models and phase diagrams
- Cosmological Membranes

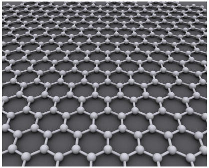
Why membranes?

Biological:



- Phospholipid bilayer
- Cytoskeleton

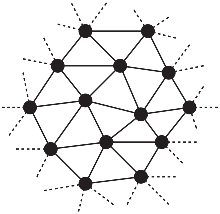
Non-biological:



- Two-dimensional crystals

Effective vs fundamental

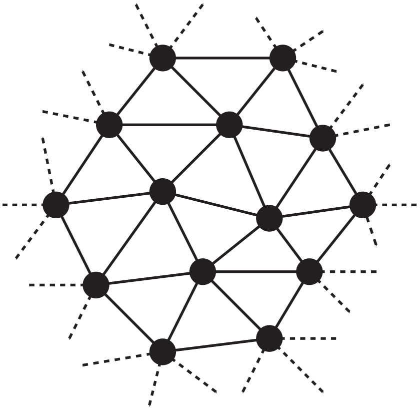
Effective:



Fundamental: String theory, Brane-models.

A microscopic model

Imagine a membrane as realized through the bonding of *monomers*:

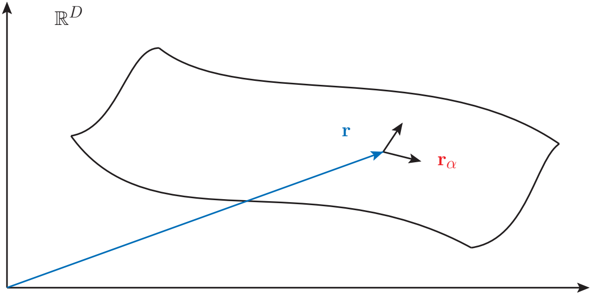


To fix a scale, let the *bonds* length be of order $1\mu m$.

A macroscopic model

At higher scales, say $\gtrsim 10\mu m - 1nm$, the membrane admits an *effective* continuous description $\mathrm{ISO}(D)$-symmetric:

$$\mathbf{r} : \mathbb{R}^d \to \mathbb{R}^D$$



The "physical" case corresponds to $D = 3$ and $d = 2$.

A bit of geometry

Induced metric:

$$g_{ab} = \partial_a r^\mu \partial_b r^\nu \delta_{\mu\nu}$$

The geometry is characterized by both intrinsic and extrinsic curvatures:

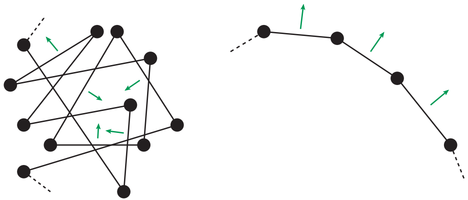
$$\partial_a \partial_b r^\mu = K^i_{ab} n^\mu_i + \Gamma_a{}^c{}_b \partial_c r^\mu$$

Framing of the normal bundle is controlled by a connection:

$$A_a{}^i{}_j = n^\mu_i \partial_a n^\nu_j \delta_{\mu\nu}$$

Order parameters I

Ising-like normals (flat vs crumpled):



The orientational order is controlled by the extrinsic curvature:

$$K^2 \sim (\partial^2 r^\mu)^2 \sim (\partial n^i)^2$$

A word on perturbation theory

Rigidity of the membrane is generally controlled by

$$\partial_\alpha \partial_\beta r^\mu$$

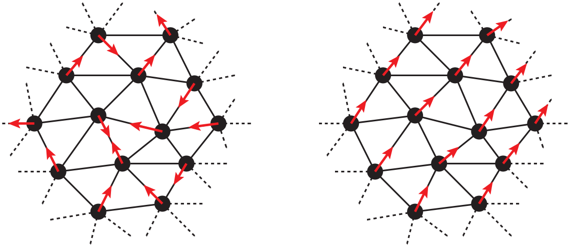
Therefore in perturbation theory and for the physical case $\epsilon = 2$, because the upper critical dimension is $d = 4$.

We need a non-perturbative method to draw phase-diagrams.

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2)} + R_k\right)^{-1} k\partial_k R_k$$

Order parameters II

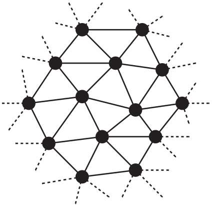
Local order:

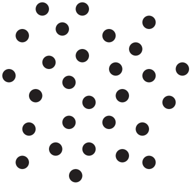


Order parameters for the breaking of the internal symmetries.
$\mathbb{Z}_6$ local order is promoted to $\mathrm{SO}(2)$ symmetry in the continuum.

Microscopic (dis)order

Our membrane is subject only to mechanical stresses and temperature T.
Microscopically, T is responsible for *melting*:





At low T we have a *crystalline* phase: *microscopic order*.

At high T the bonds melt to *fluid* phase: *microscopic disordered*.

The KTNHY description

Kosterlitz, Thouless '73; Nelson, Halperin '79; Young '79

Coset models

Embedding's isometries:

$$\mathrm{ISO}(D): \quad r^\mu \to R(\alpha)^\mu{}_\nu r^\nu + b^\mu$$

Full symmetry group:

$$\mathrm{ISO}(D) \times G_{\mathrm{int}}$$

Extension is generated breaking (at least) the translations.
Given the unbroken group $H \supset H_0$, all the membrane models are constructed as

$$\mathrm{ISO}(D) \times G_{\mathrm{int}} \big/ H_0$$

West '00

The tethered membrane model I

Each monomer breaks translations fully:

$$\mathrm{ISO}(D)\big/\mathrm{SO}(D)$$

The order parameters of the broken translations are $\partial_\alpha r^\mu$.

$$\begin{aligned} S[\mathbf{r}] &= \int \mathrm{d}^d x \left(\frac{\kappa}{2} \left(\partial_\alpha \partial_\alpha \mathbf{r}\right)^2 + \frac{t}{2} \left(\partial_\alpha \mathbf{r}\right)^2 \right. \\ &\qquad \left. + u \left(\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r}\right)^2 + v \left(\partial_\alpha \mathbf{r} \cdot \partial_\alpha \mathbf{r}\right)^2 + \ldots \right) \end{aligned}$$

- κ: bending rigidity.
- t: tension.
- u and v: Lamé coefficients.

Deformations: $r^\mu \to r^\mu + \delta r^\mu$

The tethered membrane model II

$$\Gamma_k[\mathbf{r}] = \int \mathrm{d}^d x \Big(\frac{Z_k}{2} (\partial_\alpha \partial_\alpha \mathbf{r})^2 + U_k[\partial_\alpha \mathbf{r}] \Big)$$

Up to 8th order in the derivative expansion:

- Flat phase characterized by critical exponent $\eta = 0.849$. Very good agreement with MC simulations (perturbation theory: $\eta \simeq 0.96$).
  η governs in-plane and out-of-plane deformations' scaling behavior thanks to long-range order.
- $D_{\mathrm{cr}} \simeq 4.5$ for a 2nd order PT.
  Perturbation theory suggests $D_{\mathrm{cr}} \simeq 219$.

Essafi, Kownacki, Mouhanna '14

The fluid membrane model I

Bonds melt, the infinitesimal plaquette $r^\mu + \partial_\alpha r^\mu \mathrm{d}x^\alpha$ breaks:

$$\mathrm{ISO}(D)/\mathrm{SO}(d) \times \mathrm{SO}(D-d)$$

The order parameter is a frame ${e_a}^\alpha$ with $g_{ab} = {e_a}^\alpha {e_b}^\beta \delta_{\alpha\beta}$.
Unbroken translations survive as reparametrization invariance.

$$S[\mathbf{r}] \;=\; \int \mathrm{d}^d x \sqrt{g}\left(\mu + \frac{\kappa}{2}K^2 + \frac{\bar{\kappa}}{2}R + \dots\right)$$

- κ: bending rigidity.
- μ: surface tension.
- $\bar{\kappa}$: Gaussian rigidity.

Deformations: $r^\mu \to r^\mu + \nu^a \partial_a r^\mu + \nu^i n^i \to r^\mu + \nu^i n^i$

The fluid membrane model II

$$\Gamma_k[\mathbf{r}] \;\; = \;\; \int \mathrm{d}^d x \sqrt{g} \left(\mu_k + \frac{\kappa_k}{2} K^2 + \frac{\bar{\kappa}_k}{2} R \right)$$

No evidence for a non-trivial PT for $d = 2$. The fluid membrane is always crumpled.

However, interesting relations between the fluid model and $2d$ quantum gravity.

Codello, Zanusso '11

Long–range effects from the melting of the tethered model

$$S[\mathbf{r}] \;=\; \int \mathrm{d}^2x \left(\epsilon_0 + \frac{t}{2}\,(\partial_\alpha \mathbf{r})^2 + \ldots\right)$$

Field dependent transformation to highlight the Goldstones of the broken rotations of the fluid phase:

$$\partial_\alpha r^\mu \to R(\xi)^\mu{}_\nu \partial_\alpha r^\nu$$

and functionally integrate them:

$$\Gamma[\mathbf{r}] = \int \mathrm{d}^2x \sqrt{g}\left(t_R - \frac{1}{96\pi} R \frac{1}{\Delta} R\right)$$

The fluid description of the tethered model enjoys long range interactions.

Z. '14

The hexatic membrane model

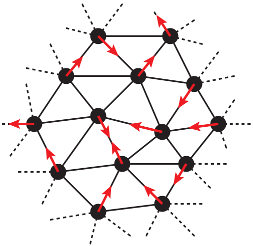
Crystalline structure breaks internal rotations:

$$\mathrm{ISO}(D) \times \mathrm{SO}(d)_{\mathrm{cr}} / \mathrm{SO}(d)_{\mathrm{diag}} \times \mathrm{SO}(D-d)$$

New order parameter $N^\alpha = \cos\theta\, {e_1}^\alpha + \sin\theta\, {e_2}^\alpha$ (d=2).

$$S[\mathbf{r}, \mathbf{N}] \;=\; \int \mathrm{d}^d x \sqrt{g} \Big(\mu + \frac{\kappa}{2} K^2 + \frac{\bar{\kappa}}{2} R + \frac{K_A}{2} (\nabla_a \mathbf{N})^2 + \ldots \Big)$$

- K_A: Hexatic rigidity.



David, Guitter, Peliti '87; Park, Lubensky '96; ...

David's strategy

$N^\alpha = \cos\theta\, {e_1}^\alpha + \sin\theta\, {e_2}^\alpha$:

$$S[\theta, \mathbf{r}] = \int d^2x \sqrt{g} \left\{ \mu + \frac{\kappa}{2} K^2 + \frac{\bar{\kappa}}{2} R + \frac{K_A}{2} (\partial_a \theta + \omega_a)^2 \right\}$$

Complete the square $\theta \to \theta - \int \frac{1}{\Delta} \nabla_a \omega^a$:

$$S[\theta, \mathbf{r}] = \int d^2x \sqrt{g} \left\{ \mu + \frac{\kappa}{2} K^2 + \frac{\bar{\kappa}}{2} R + \frac{K_A}{2} (\partial_a \theta)^2 + \frac{K_A}{8} R \frac{1}{\Delta} R \right\}$$

θ is integrated away (finite renormalization for K_A):

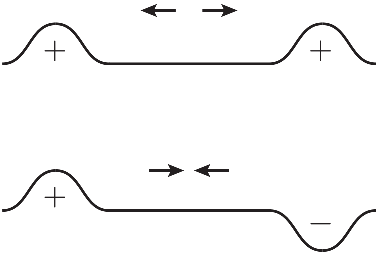
$$\Gamma[\mathbf{r}] \;\; = \;\; \int \mathrm{d}^d x \sqrt{g} \Big(\mu + \frac{\kappa}{2} K^2 + \frac{\bar{\kappa}}{2} R + \frac{K_A}{8} R \frac{1}{\Delta} R + \dots \Big)$$

David '89

Long–range effects (again)

N induces long range interactions among curvatures.
Analog effects are generated when integrating d.o.f. from tethered to fluid models.

(Very) schematically:



The Gaussian curvature is source for a Kosterlitz–Thouless type transition. Free from Mermin–Wagner theorem.

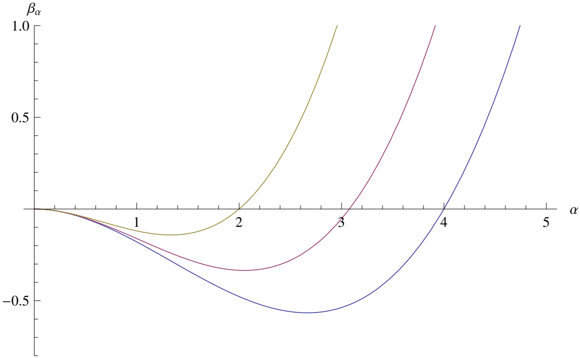
Beta functions in the FRG scheme, $\alpha = 1/\kappa$

$$
\begin{aligned}
k\partial_k\tilde{\mu}_k &= -2\tilde{\mu}_k - \frac{D-2}{2\pi\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}\log\frac{2+\tilde{\mu}_k-\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}{2+\tilde{\mu}_k+\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}} \\
k\partial_k\alpha_k &= \frac{\alpha_k\left(D-\frac{3}{4}K_A\alpha_k\right)}{2\pi\left(4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}\right)}\left[\frac{2(1-\alpha_k)+\alpha_k\tilde{\mu}_k}{1+\alpha_k\tilde{\mu}_k}\right. \\
&\quad \left. + \frac{\tilde{\mu}_k}{\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}\log\frac{2+\tilde{\mu}_k-\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}{2+\tilde{\mu}_k+\sqrt{4+\tilde{\mu}_k^2-\frac{4}{\alpha_k}}}\right]
\end{aligned}
$$

Codello, Z. '13

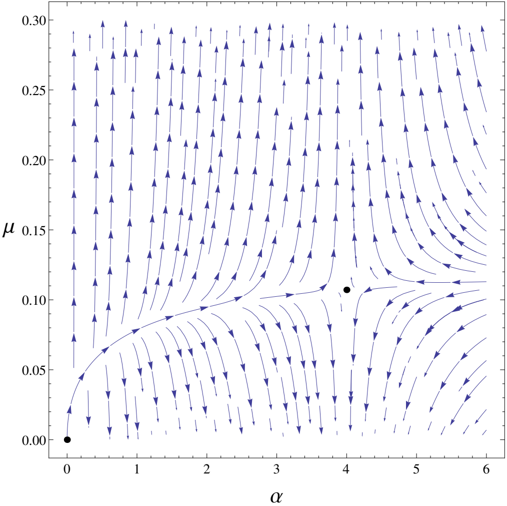
A 2nd order PT in $\alpha = 1/\kappa$

Non-trivial fixed point $\alpha^* \sim 1/K_A$.
Critical exponent: $\nu \simeq 0.37$.
Beta function of α for $K_A = 1/2, 1, 2$ from bottom:



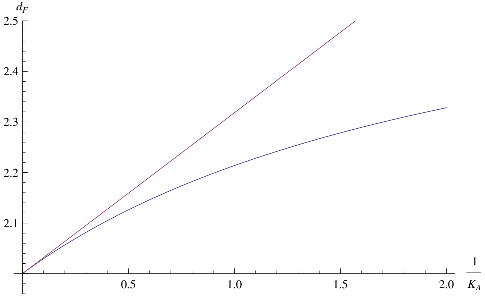
Drawing the phase–diagram

For $K_A = 1$:



It is extended but not quite

By decreasing K_A the fractal dimension saturates to 2.71 in our estimate.



Crinkled phase: the long–range interactions stack positive and negative curvatures together making the (ground state) surface very fuzzy.

Cosmology and the Galileon

Galileon theory with symmetry $\pi(x) \to \pi(x) + b_\alpha x^\alpha + c$:

$$S[\pi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\pi)^2 - \frac{\nu}{2}(\partial\pi)^2 \Box\pi + \dots \right)$$

Interpreted as a DGP brane model $g_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\alpha\pi\partial_\beta\pi$ in the non-relativistic limit $(\partial\pi)^2 \ll 1$.

In general:

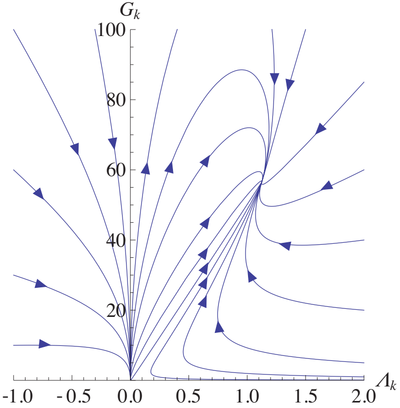
$$S[\pi] = \int d^d x \sqrt{g} \left(\mu + \nu K + \frac{\kappa}{2} K^2 + \frac{\bar{\kappa}}{2} R \right)$$

The brane is embedded in $D = d + 1$ dimensions.

de Rham, Tolley '10; Brouzakis, Codello, Tetradis, Z. '13

Non-perturbative renormalization of the Galileon

$$\mu = \frac{\Lambda}{8\pi G} \qquad \bar{\kappa} = -\frac{1}{8\pi G}$$



Codello, Tetradis, Z. '12

Conclusions

- $2d$ extended objects: microscopic order $\Rightarrow$ macroscopic phase.
- Melting of a crystalline surface is driven by long-range interactions to a non-trivial phase.
- Rich physical content.
- Much more phenomenology to be captured.

Prospects

- Defects/disinclinations.
- Self-avoidance.