Asymptotic Safety of nonlinear O(N)-Models

A. Wipf

Theoretisch-Physikalisches Institut, FSU Jena

collaboration with

FRG: Raphael Flore and Omar Zanusso
Lattice: Daniel Körner and Bjoern Wellegehausen

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Asymptotic Safety

- nontrivial renormalization UV-fixed point
  with small number of IR-relevant (UV-attractive) directions
- UV-completion for scalar field theory — K. Wilson and J. Kogut, 1974
- Quantum Gravity could be asymptotically safe — S. Weinberg, 1976
  accumulating evidence in past years, based on FRG

Conjecture: renormalizable and asymptotically free in d dimensions
$\Rightarrow$ asymptotic safe in $d+\epsilon$ dimensions — S. Weinberg

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- gravity in $2+\epsilon$ dimensions — S. Weinberg, 1979
- large-N Gross-Neveu model in 3 dimensions — Gawedzki and A. Kupiainen, 1985
- 4-Fermi models in 3 dimensions (FRG) — Janssen, Gies
- linear O(N) models in $2 \leq d < 4$ dimensions — Wetterich, Canet et al., Litim, Zappala
- gauge theories in $d > 4$ dimensions? — lattice: Raggio, Knechtli
- nonlinear O(N) models in 3 dimensions (FRG)
  - background field method, covariant quantisation, …
    Codello, Percacci, Zanusso, Fabbrichesi, Tonero
  - covariant fourth-order calculations — R. Flore, O. Zanusso, aw 2013
- approximations $\Longrightarrow$

support from different nonperturbative methods

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Nonlinear O(N)-Models

- target-space = sphere: $\boldsymbol{n} \in \mathbb{R}^N$ and $\boldsymbol{n} \cdot \boldsymbol{n} = 1$
- classical action

$$S = \int \mathrm{d}^d x \, \partial_\mu \boldsymbol{n}(x) \cdot \partial^\mu \boldsymbol{n}(x)$$

- $N = 4$: effective model for chiral phase transition of (2+1) QCD
- $N = 3$: Heisenberg model for ferromagnetism
- $N = 2$: shows Kosterlitz-Thouless PT in $d = 2$
- $N = 1$: ubiquitous Ising model
- $d = 2$: perturbatively renormalizable
  O(3)-model is toy model for *QCD*: asymptotic freedom, instantons, dynamical mass-generation, . . .

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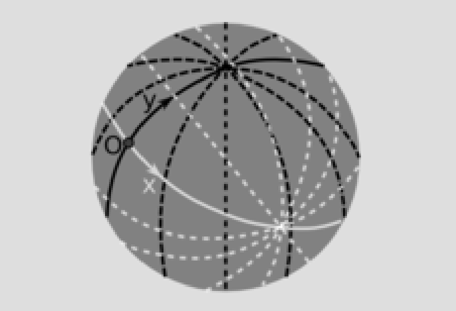
FRG for nonlinear O(N) models

- gemetric formulation: nonlinear curved target space (cp. gravity)
  split $\phi = \varphi + \xi$ has no *geometric* meaning
  instead: $\phi(\varphi, \xi)$ with geometric objects φ and ξ (cp. gravity)
- configuration space $\mathcal{M} \equiv \{\varphi : \mathbb{R}^d \to S^{N-1}\}$

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Geometry of target space

- covariant *classical* action

$$S[\varphi] = \frac{\zeta}{2}\int \mathrm{d}^d x\; h_{ab}(\varphi)\partial_\mu\varphi^a\partial^\mu\varphi^b$$

- Ricci curvature $R_{ab} = (N-2)h_{ab} \rightarrow$ dependence on N
- pull back of covariant derivative

$$\nabla_\mu v^a \equiv \partial_\mu v^a + \partial_\mu\varphi^b\Gamma_b{}^a{}_c v^c \Rightarrow \triangle$$

- $\sim$ *Vilkovisky-connection* in gravity and gauge theories

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Covariant formulation

- geodesic $\varphi(s)$ from background-field φ to field ϕ
- $\varphi(s=0)=\varphi$ and $\varphi(s=1)=\phi$
- tangent vectors $\xi^a(s)$, set $\xi^a(0)=\xi^a$
- derivative along geodesic $\nabla_s=\xi^a(s)\nabla_a$
- functional of field — Mukhi

$$F[\phi]=\sum_{n\geq 0}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}F[\varphi(s)]\Big|_{s=0}=\sum_{n\geq 0}\frac{1}{n!}\nabla_s^n F[\varphi(s)]\Big|_{s=0}$$

$$=\sum_{n\geq 0}F^n_{(a_1,\dots,a_n)}[\varphi]\,\xi^{a_1}\cdots\xi^{a_n}=F[\varphi,\xi]$$

- $F^n_{(a_1,\dots,a_n)}$ in terms of curvature and its covariant derivatives

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- local action up to fourth order in derivatives

$$\Gamma^{\mathrm{s}}_k[\phi] = \frac{1}{2}\int d^d x\, \Big(\zeta_k h_{ab}\partial_\mu\phi^a\partial^\mu\phi^b + \alpha_k h_{ab}\Delta\phi^a\Delta\phi^b$$

$$+\, T_{abcd}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b\partial_\nu\phi^c\partial^\nu\phi^d\Big)\Gamma_k[\phi(\varphi,\xi)]$$

$$T_{abcd} = L_{1,k} h_{a(c}h_{d)b} + L_{2,k} h_{ab}h_{cd}$$

- wave function renormalization for vector fluctuation

$$\Gamma_k[\varphi,\xi] = \Gamma^{\mathrm{s}}_k[\phi(\varphi, Z_k^{1/2}\xi)]$$

- bi-field cutoff action

$$\Delta S_k[\varphi,\xi] = \frac{1}{2}\int \mathrm{d}^d x\, \xi^a \mathcal{R}^k_{ab}[\varphi]\xi^b$$

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- cutoff operator $\mathcal{R}^k_{ab}[\varphi] = Z_k h_{ab} R_k[\Delta]$
- plug into flow equation

$$k\partial_k \Gamma_k[\varphi,\xi] = \frac{1}{2}\mathrm{Tr}\left(\frac{k\partial_k \mathcal{R}_k[\varphi]}{\Gamma_k^{(0,2)}[\varphi,\xi] + \mathcal{R}_k[\varphi])}\right) .$$

- $\Gamma_k^{(0,2)}[\varphi,0]$ is 4^{th} order elliptic differential operator
- adapted 4^{th} order cutoff-operator

$$R_k(\Delta) = \left[\zeta_k(k^2-\Delta) + \alpha_k(k^2-\Delta^4)\right]\theta(k^2-\Delta)$$

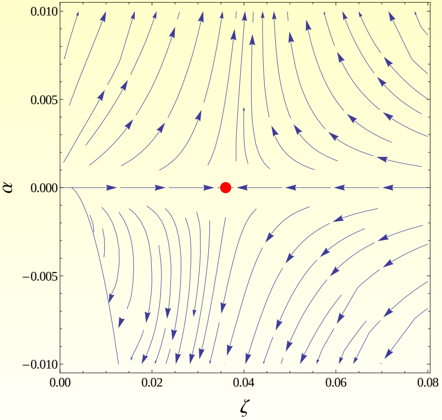
$\Rightarrow$ off-diagonal heat kernel method

- heat kernel expansion, Laplace/Mellin-transform

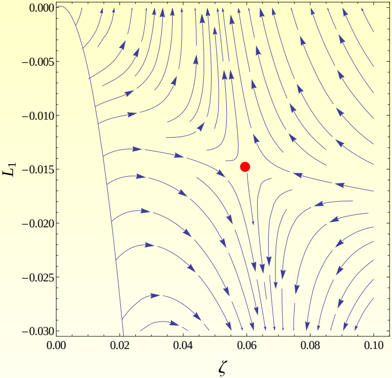
$\Rightarrow$ flow of couplings, β-functions, anomalous dimension, ...

Some results

- two coupling O(3)-model has fixed point in (ζ_k, α_k)-plane
- arrows point to UV



- three coupling O(3)-model has fixed point in (ζ_k, α_k, L_1)-space
- slice $\alpha = 0$, one irrelevant direction



R. Flore, O. Zanusso, aw Phys.Rev. D87 (2013) 065019

- critical exponent $\nu(N)$ can be estimated
- 4^{th} operator $\propto L_2$: *fixed points disappears*

  $h_{ab}h_{cd}\partial_\mu\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\nu\phi^d$
- same with exp. regulator
- reappears in higher order truncation (phase space flow)

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Nonlinear O(N) Models on Lattice

- $\boldsymbol{n}^2 = 1 \Rightarrow$ derivative expansion for effective action

$$S[\boldsymbol{n}] = \sum_{\alpha=0}^{3} g_\alpha N S_\alpha[\boldsymbol{n}] + \mathcal{O}(\partial^6), \quad \boldsymbol{n}^2 = 1$$

- all operators with $\leq$ four derivatives

$$S_0 = -\int \mathrm{d}^d x \; \boldsymbol{n} \cdot \Delta \boldsymbol{n}$$

$$S_1 = \int \mathrm{d}^d x \; \boldsymbol{n} \cdot \Delta^2 \boldsymbol{n}$$

$$S_2 = \int \mathrm{d}^d x \; (\boldsymbol{n} \cdot \Delta \boldsymbol{n})^2$$

$$S_3 = \int \mathrm{d}^d x \; (\boldsymbol{n} \cdot \partial_\mu \partial^\nu \boldsymbol{n})(\boldsymbol{n} \cdot \partial^\mu \partial_\nu \boldsymbol{n})$$

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- standard discretization
- RG transformation: field n on fine grid $(N, a) \Longrightarrow$ averaged field n' on coarser grid $(N' = N/b, a' = ba)$
- physical IR-cutoff is fixed, UV-cutoff lowered $\Lambda \to \Lambda' = \Lambda/b$
- $g_\alpha \to g'_\alpha$ due to quantum fluctuation with scales in $[\Lambda', \Lambda]$
- blockspin transformation: draw averaged field n' according to

$$\mathcal{P}(\boldsymbol{n}'_x) \propto \exp\left(C(g_\alpha)\, \boldsymbol{n}'_x \cdot \sum_{y \in \Box_x} \boldsymbol{n}_y \right)$$

- $C(g_\alpha) > 0$ such that truncation errors in MCRG are minimized

Monte Carlo Renormalization Group (MCRG)

- classical MCRG: define blocked observables
  - blockspin transformation with blocking kernel …
  - localize fixed point
  - linearize MCRG-transformation in vicinity of fixed point
    $\Rightarrow$ critical exponents
- want to compute (truncated) effective action
- inverse MC + Schwinger-Dyson eqs. or demon method

M. Creutz, Hasenbusch et al., Jena group

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Demon method

- efficient method to measure couplings β_α
- (truncated) action of system ($\propto$ heat bath)

$$S_{\rm Sys} = \sum_{\alpha=1}^{n} \beta_\alpha S_\alpha$$

- add n auxiliary demons (thermometers) with action

$$S_{\rm D} = \sum_\alpha \beta_\alpha E^n_{\alpha=1}, \quad E_\alpha \in [0, E_{\rm max}]$$

- partition function of joint systems

$$Z_{\rm total} = \int_0^{E_{\rm max}} \prod_\alpha {\rm d}E_\alpha \int \mathcal{D}\phi \, {\rm e}^{-S_{\rm Sys} - S_{\rm D}}$$

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- factorization $\Rightarrow$ averages of "Demon-energies"

$$\langle E_\alpha \rangle = -\frac{\mathrm{d}}{\mathrm{d}\beta_\alpha} \log \int_0^{E_{\max}} \mathrm{d}E \, \mathrm{e}^{-\beta_\alpha E}$$
$$= \frac{1}{\beta_\alpha} - \frac{E_{\max}}{\mathrm{e}^{\beta_\alpha E_{\max}} - 1} \approx \frac{1}{\beta_\alpha}$$

- $\langle E_\alpha \rangle$ from simulations $\Rightarrow \beta_\alpha$

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implementation of RG transformation

- generate configuration in equilibrium on fine grid for some $\{\beta_\alpha\}$
- blocking of configuration $\Rightarrow$ configuration on coarser grid distributed with $\mathrm{e}^{-S_{\mathrm{Sys}}}$ for some $\{\beta'_\alpha\}$
- microcanonical simulation of joint system on coarser grid
  - begin with blocked configuration and demon energies extracted from previous runs
  - calculate $\langle E_\alpha \rangle \Longrightarrow \beta'_\alpha$
- generate configuration on fine grid accoring for $\{\beta'_\alpha\}$
- blocking of this configuration ...

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Some interesting algorithmic aspects

- assume linear dependence $C(g) = \sum c_\alpha g_\alpha$ ($\sim$ regulator)
- truncation: in general $\xi' \neq \xi/b$ (b decimation length)

fine tune c_α such that $\xi' \approx \xi/b$

- c_α depend on couplings and N!
- near nontrivial fixed point: $g_1, g_2, g_3 \ll g_0$
  $\Rightarrow$ only fine tuning of c_0 necessary
- not so near Gaussian fixed point
- optimal choice: difficult in $d = 2$
- more robust in 3 dimensions

Körner, Wellegehausen, Wipf

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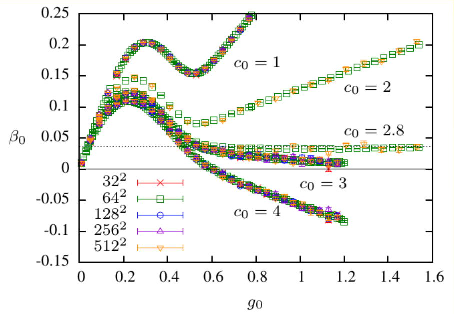
β-function in 2 dimensions

- optimal $c_0 = 2.8 \Rightarrow$
  β-function tends to large-N result $\log(2)/(6\pi)$
- $c_0 < 2.8$: one fixed point at vanishing coupling
- $c_0 > 2.8$: additional fixed point at finite coupling: truncation artifact



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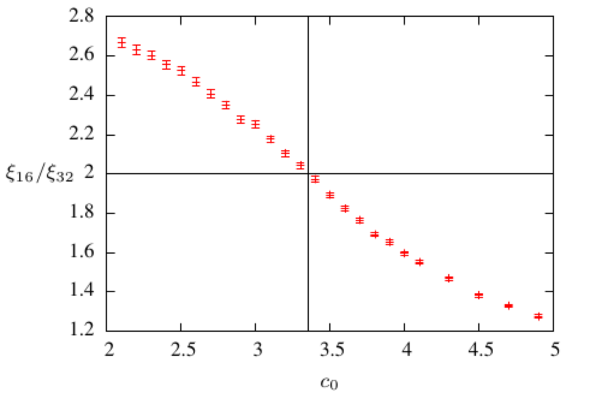
O(3) model in 3 dimensions

- simple truncation $S = (g_0 N) S_0$; blocking $32^3 \to 16^3$ sufficient
- on fine grid: $g_0^c = 0.22975(25)$
- thermodynamic limit: $g_0^c = 0.2287462(7)$ Campostrini et al. 2002)

$N = 3$:

$c_0^{\text{pert}} = 2.30$

$c_0^{\text{opt}} = 3.35$



Nontrivial FP in simple truncation Ng_0S_0

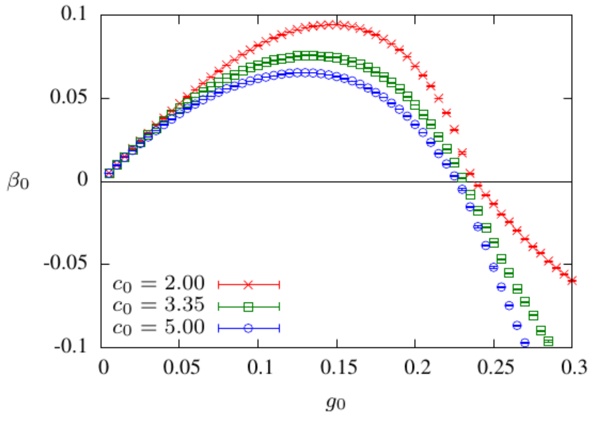
nontrivial fixed point “for all” c_0

$\beta(g_0^*) = 0$

$g_0^* = 0.2310(5)$

$g_0 < g_0^* \rightarrow$ disordered GFP

$g_0 > g_0^* \rightarrow$ ordered phase



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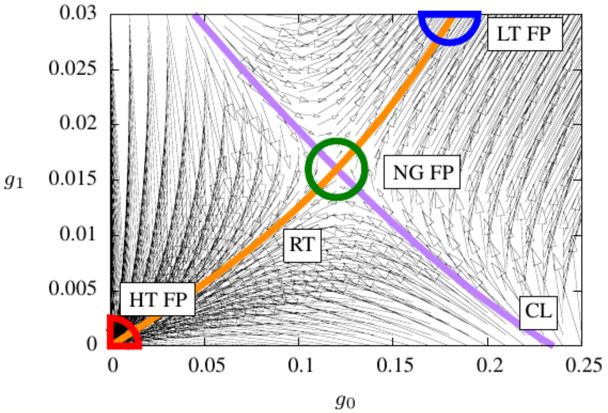
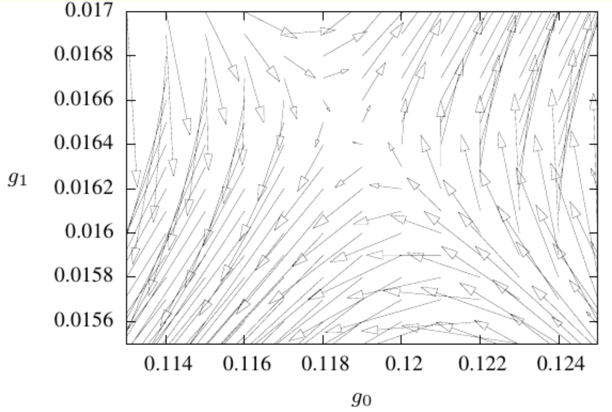
Include higher-order derivative terms

- fixed point stable?
- number of relevant directions?
- $2 \to 2$ truncation: $c_0 = 3.1$ and $c_1 = 2.5$



Vicinity of UV fixed point

- $2 \to 2$ truncation: $g_0^* = 0.119(1)$ and $g_1^* = 0.0164(2)$
- position of fixed points almost volume-independent



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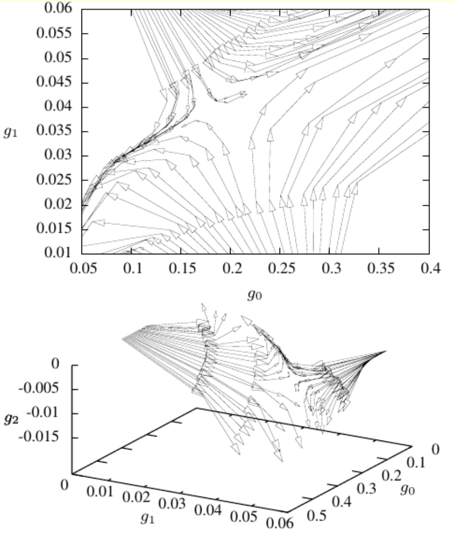
truncation with S_0, S_1, S_2

asymptotic safety:
UV-attractive
(IR-relevant)
directions
exhausted



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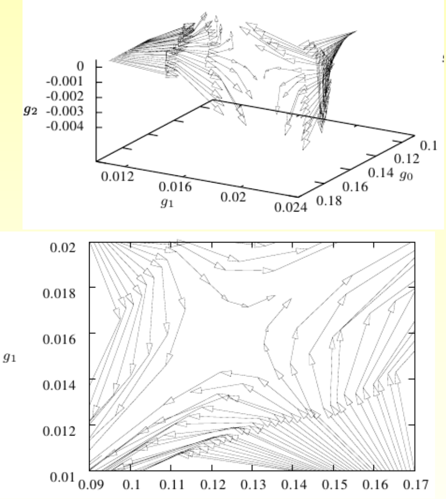
detailed picture of S_0, S_1, S_2 truncation

one UV-attractive direction

$$g_0^* = 0.13(1)$$
$$g_1^* = 0.016(1)(1)$$
$$g_2^* = -0.0015(5)$$



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detailed picture of S_0, S_1, S_3 truncation

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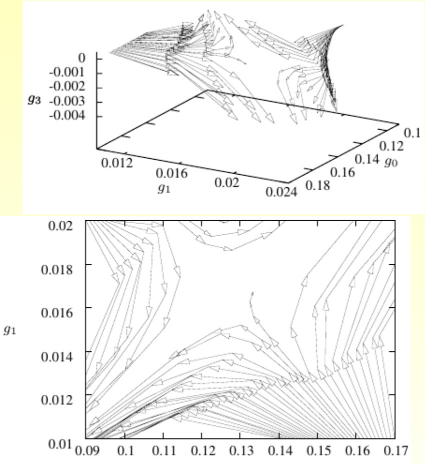

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detailed picture of S_0, S_1, S_2, S_3 truncation

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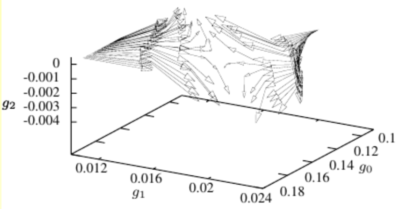

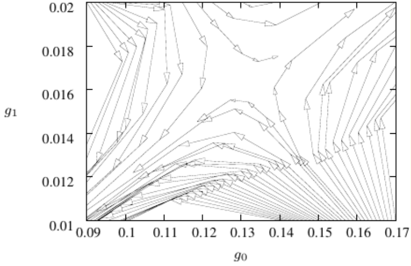

No instabilities as in FRG-approach seen!

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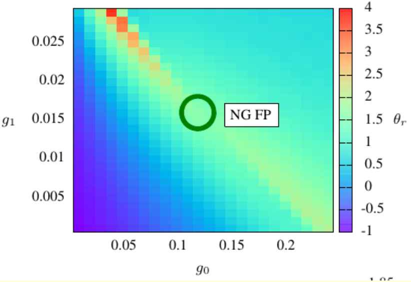
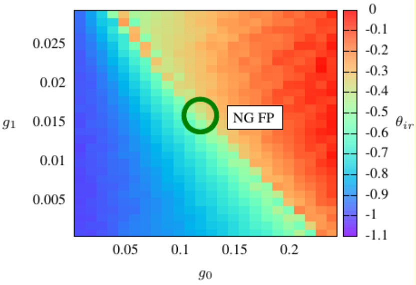
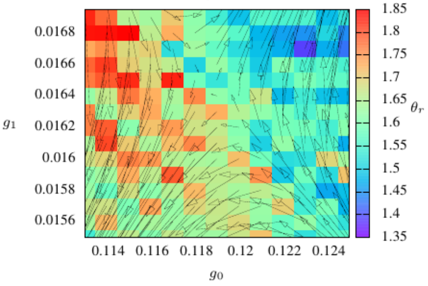
Critical exponents







upper left $\theta_{\text{rel}} = 1.61(4)$

$$\nu = 0.62(3)$$

$$\nu_{\text{true}} = 0.7112(5)$$

upper right $\theta_{\text{irr}} = -0.44$

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Results for critical exponent ν

Method	ν	ν/ν_{MCHT}
$1 \to 1$ trunc. ($c_0 = 3.35$)	0.51(1)	~ 0.72
$1 \to 2$ trunc. ($c_0 = 3.35$)	0.55(2)	~ 0.77
$2 \to 2$ trunc. ($c_0 = 3.1,\ c_1 = 2.5$)	0.62(3)	~ 0.87
$2 \to 2$ trunc. ($c_0 = 3.4,\ c_1 = 1.0$)	0.66(4)	~ 0.93
$3 \to 3$ trunc. ($c_0 = 3.1,\ c_1 = 2.5,\ c_2 = 0$)	0.64(3)	~ 0.90
FRG	0.704	~ 0.99
MCHT	0.7112(5)	1
MC	0.7116(10)	~ 1
RG	0.706	~ 0.99
HT	0.715(3)	~ 1

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Dependence on N

- optimal c_α in blocking kernel depends on N
- qualitatively similar flow diagrams for $N = 3, 4, 5, \ldots$
- position of UV-FP varies with N
- critical exponents: comparison with large-N expansions
  comparison with RG-approach

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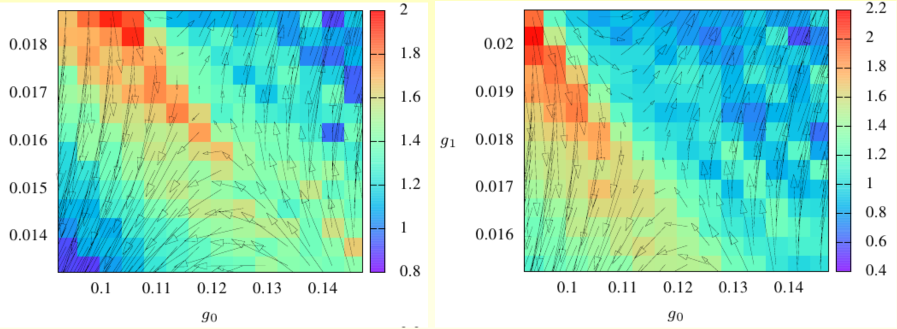
Flow diagrams and critical exponents for $N > 3$



O(4)-model O(6)-model .

Critical exponent ν

N	2	3	4	5	6
$1 \rightarrow 1$ truncation	0.42	0.51	0.57	0.63	0.65
$2 \rightarrow 2$ truncation	0.64(4)	0.66(4)	0.71(5)	0.78(6)	0.81(6)
FRG	-	0.704	0.833	-	0.895
HT exp.	0.677(3)	0.715(3)	0.750(3)	-	0.804(3)
RG exp.	0.607	0.706	0.738	0.766	0.790

N	7	8	9	10
$1 \rightarrow 1$ truncation	0.68	0.65	0.62	0.58
$2 \rightarrow 2$ truncation	0.86(7)	0.84(7)	0.89(8)	
FRG	-	0.912	-	0.920
HT exp.	-	0.840(3)	-	0.867(4)
RG exp.	0.811	0.830	0.845	0.859

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Conclusion

- FRG detects nontrivial UV fixed point for every N
- stability problem of FRG with one 4^{th}-order operator (background-field method? spectral adjustment?)
- fixed point reappears with inclusions of higher derivatives
- in MCRG approach no stability problems (different truncation)
- FRG and MCRG in qualitative agreement
- results sufficient for global picture on flow diagram

existence UV-fixed point for $3 \leq N \leq 15$ firmly established
by FRG and stable lattice results

thanks, also to Nikos

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