Scalar-fermions systems in LPA approximation: a two function truncation

Gian Paolo Vacca

INFN - Bologna

Work in collaboration with Luca Zambelli
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Introduction

QFT of one real scalar field and Dirac fermions (N_f, d_γ) $X_f = N_f\, d_\gamma$

Symmetries: $U(N_f)$ Z_2

Typically the lagrangian density one considers is: $\mathcal{L} = \frac{1}{2}\,\partial^\mu\phi\partial_\mu\phi + V(\phi) + \bar{\psi}\gamma^\mu i\partial_\mu\psi + i\, y\, \phi\, \bar{\psi}\, \psi$

and a polynomial form of the scalar potential is used.

Gies, Scherer 2010 (d=4) , Rosa, Vitale, Wetterich 2001 , Sonoda 2011 (d=3) , …..

For a SUSY model both the scalar potential and Yukawa interactions in the on shell lagrangian density are dependent on the same function. A study in term of polynomial truncations has been done.

Synatschke, Braun, Wipf 2010 (d=3)

$$\mathcal{L}_{\rm on} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}W'^2(\phi) - \frac{1}{2}W''(\phi)\bar{\psi}\psi$$

Here we shall study this QFT using the following truncation where we neglect terms related to 2n fermion interactions (n>1) which may exist depending on (N_f, d_γ)

$$\Gamma_k[\phi,\psi,\bar{\psi}] = \int d^4x \left(\frac{1}{2} Z_{\phi,k}\, \partial^\mu\phi\partial_\mu\phi + V_k(\phi) + Z_{\psi,k}\bar{\psi}\gamma^\mu i\partial_\mu\psi + i\, H_k(\phi)\, \bar{\psi}\, \psi \right)$$

and essentially limiting ourself to the LPA (Z's=1)

A very recent work for quark-mesonic interactions also considers more general Yukawa interactions

Pawlowski, Rennecke 1403.1179

Introduction

Qualitative observations

- Critical dimensions: from field's canonical dimensions $d_\phi = \frac{d}{2} - 1$ $\quad d_\psi = \frac{d-1}{2}$

  Depending on the dimension there are several possible relevant operators

$$\phi^{2n} \qquad d_c^{(v)}(n \geq 2) = \frac{2n}{n-1} = 4\,,\, 3\,,\, \frac{8}{3}\,,\, \frac{5}{2}\,,\, \frac{12}{5}\,,\, \cdots$$

$$\phi^{2n+1}\bar{\psi}\psi \qquad d_c^{(h)}(n \geq 0) = \frac{4(n+1)}{2n+1} = 4\,,\, \frac{8}{3}\,,\, \frac{12}{5}\,,\, \cdots$$

- Classical scaling behaviour at the fixed point

  For dimensionless quantities the leading asymptotic behaviour ($|\phi| \to \infty$) is determined by

$$0 = -dv + \frac{d-2+\eta_\phi}{2}\phi\, v' \qquad 0 = (\eta_\psi - 1)\, h + \frac{d-2+\eta_\phi}{2}\phi\, h'$$

  In the LPA since $\eta_\phi = \eta_\psi = 0$ $\qquad v \sim \phi^{\frac{2d}{d-2}} \qquad h \sim \phi^{\frac{2}{d-2}}$

  Only for d=4 the linear truncation $h(\phi) = y\,\phi$ has the correct behaviour at $|\phi| \to \infty$

Functional RG approach

We shall study the flow of the Euclidean Average Effective Action Wetterich

- In terms of a field multiplet Φ containing bosonic and fermionic d.o.f. the flow equation reads

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)}[\Phi] + \hat{R}_k\right)\partial_t \hat{R}_k\right]$$

$$\Phi^T = (\phi, \psi^T, \bar{\psi})$$

$$\partial_t = k\partial_k$$

where through the cutoff operator $\hat{R}_k$ one can control the contributions of the low scale fluctuations in the functional integral.

- We employ the optimised cutoff [Litim].

$$R_k^\phi(z) = k^2 r(z/k^2)$$

For the scalar field: $R_k^\phi(-\partial^2)$

$$r(y) = (1-y)\theta(1-y)$$

For the fermion fields: $R_k^\psi(i\not\partial) = (\sqrt{P_k(-\partial^2)/(-\partial^2)} - 1)i\not\partial$

$$P_k(z) = z + k^2 r(z/k^2)$$

- Moving to dimensionless quantities (LPA case) and renaming $\tilde{\phi} \to \phi$ one gets…

$$\tilde{\phi} = \phi\, k^{1-d/2}$$

$$v(\tilde{\phi}) = V(\phi)\, k^{-d}$$

$$h(\tilde{\phi}) = H(\phi)/k$$

See also Zanusso, Zambelli, Vacca, Percacci 2010

LPA flow equations:

$$C_d^{-1} = (4\pi)^{d/2}\Gamma\left(1+\frac{d}{2}\right)$$

$$\dot v = -dv + \frac{d-2}{2}\phi\, v' + C_d\left(\frac{1}{1+v''} - \frac{X_f}{1+h^2}\right)$$

$$\dot h = -h + \frac{d-2}{2}\phi\, h' + C_d\left[2h\,(h')^2\left(\frac{1}{(1+h^2)^2\,(1+v'')} + \frac{1}{(1+h^2)\,(1+v'')^2}\right) - \frac{h''}{(1+v'')^2}\right]$$

Symmetries: v even and h odd $\qquad v'(0)=0 \qquad$ h(0)=0

Parameters of the problem: $(d\,,\,X_f)$

Fixed points: search for scaling solutions $\quad \dot v = \dot h = 0$

No analytic solutions.

Strategy:

- Numerical evolution from the origin — Morris, Codello, …
- Numerical evolution from the asymptotic region — Morris
- Polynomial truncations — Everyone

Numerical evolution from the origin

We provide the boundary (initial condition): $v'(0) = 0 \quad , \quad h(0) = 0$ and $\boxed{v''(0) = \sigma \quad , \quad h'(0) = h_1}$

Let us show how far a numerical resolutor can evolve from the origin before encountering a singularity.

Example with $X_f = 1$

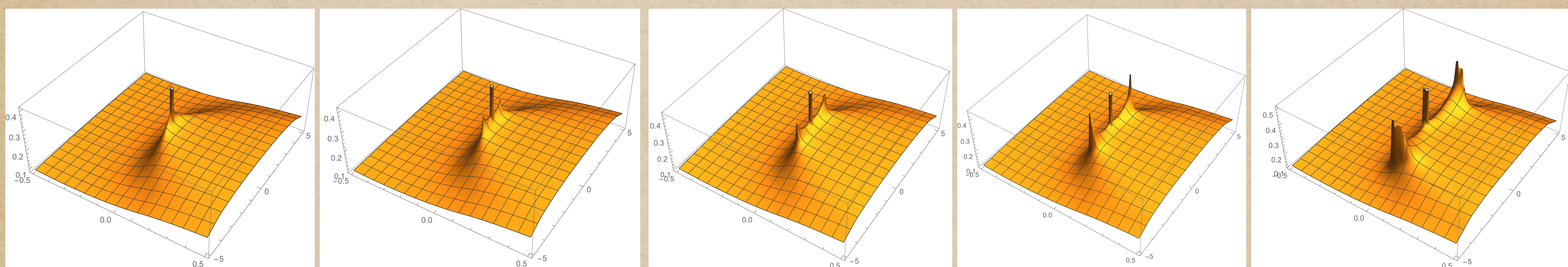
$$\frac{5}{2} < \frac{8}{3} < 3 < 4$$

$$\longleftarrow d$$

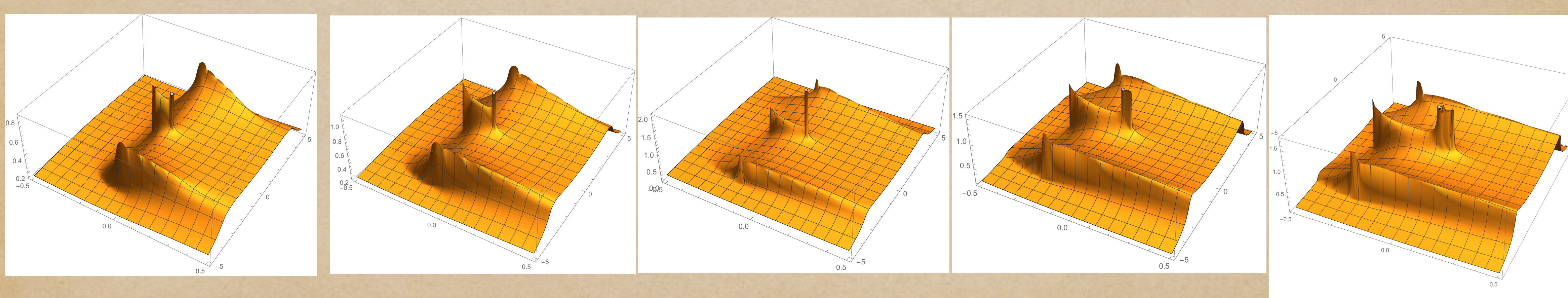
$$-0.5 < \sigma < 0.5$$

$$-5 < h_1 < 5$$

4 and 8/3 are critical for both potentials, 3 and 5/2 are critical only for the scalar potential.



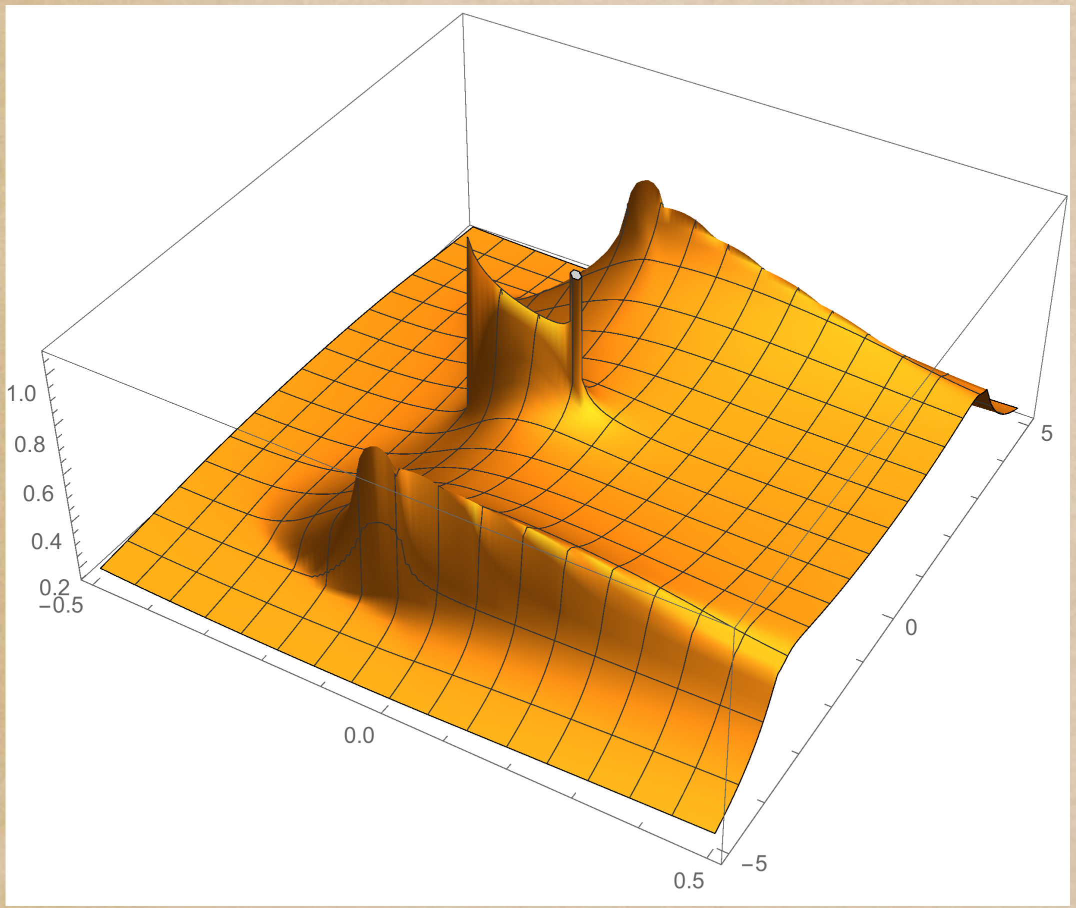
d: 4 3.98 3.9 3.75 3.5



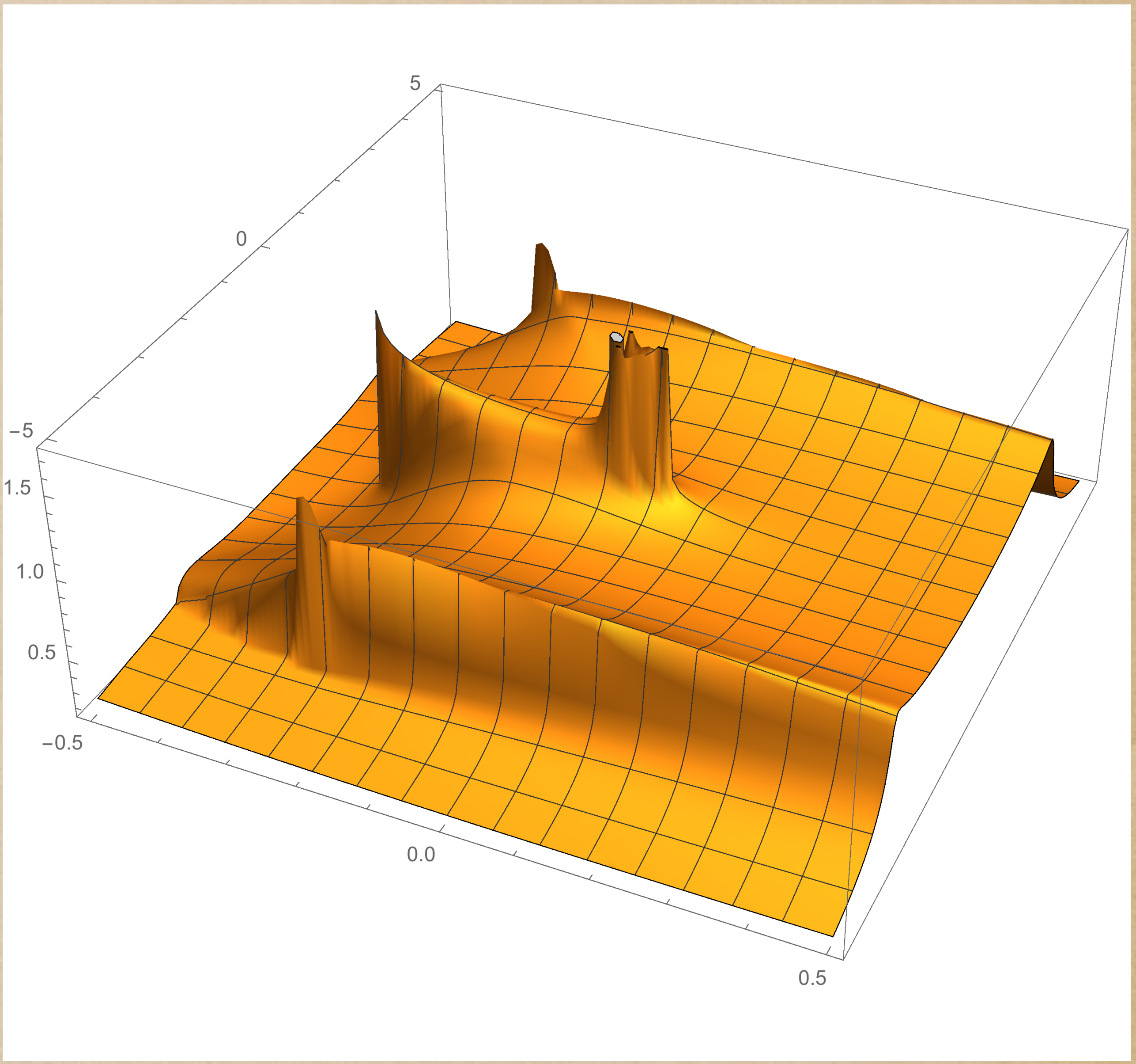
d: 3.25 3 2.9 8/3 2.57

Numerical evolution from the origin

Let's zoom in two of them

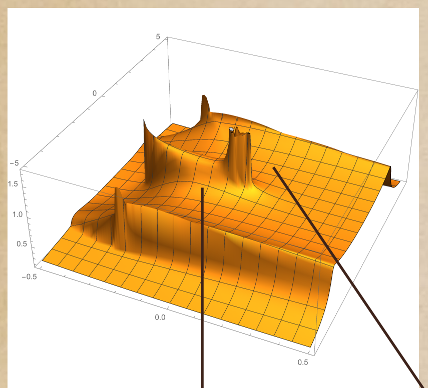


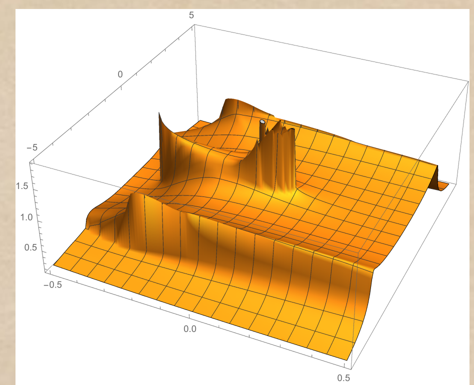
d=3



d=2.57 $< \frac{8}{3}$

Numerical evolution from the origin

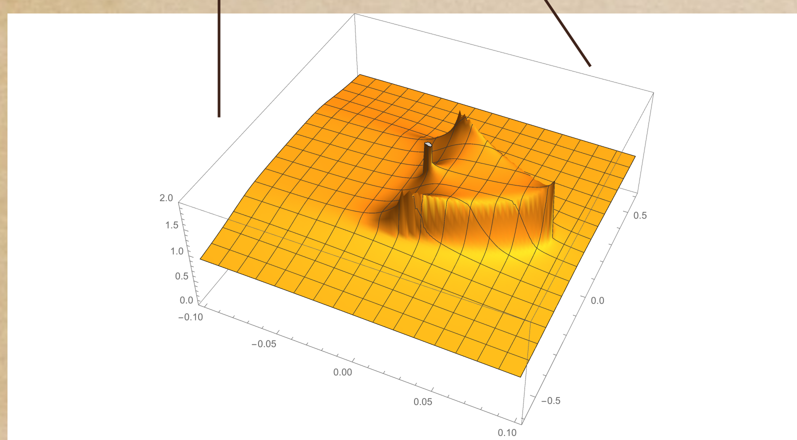




$-0.5 < \sigma < 0.5$

$-5 < h_1 < 5$

d: 2.57 2.51

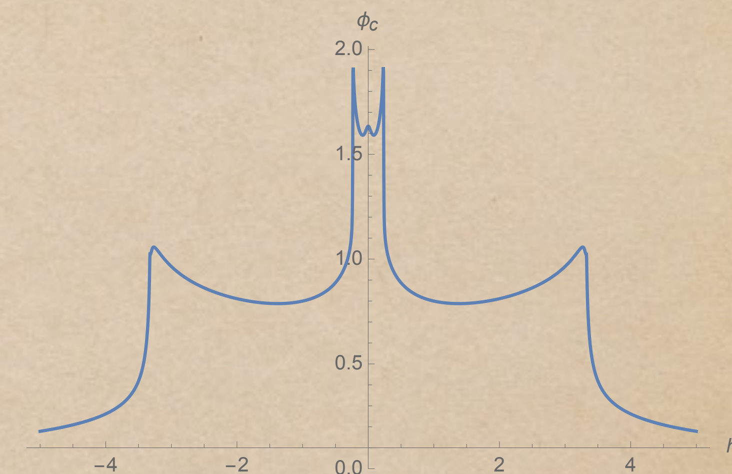


$-0.1 < \sigma < 0.1$

$-0.5 < h_1 < 0.5$

d: 2.57

Section for d=2.57 at constant σ
a new possible solution appears
for d<8/3



From this kind of plots a sharp peak can be a signal of the presence of a possible global solution.
Other approaches can confirm or reject such a fact.

All known scaling solutions of the pure scalar theory for a continuous d are visible.

Numerical evolution from the asymptotic region

The initial conditions are determined by the asymptotic behaviour ($|\phi| \to \infty$) for the fixed point solution of the system of ODE's

$$v_{\rm asympt}(\phi) \simeq A\,\phi^{2\alpha+2} + \phi^{-2\alpha}\frac{C_d\,(B - 2AX_f(\alpha+1)(2\alpha+1))}{(\alpha+1)(2\alpha+1)2AB(2+d)} + \cdots$$

$$\alpha = \frac{2}{d-2}$$

$$h^2_{\rm asympt}(\phi) \simeq B\,\phi^{2\alpha} + \phi^{-2-2\alpha}\frac{C_d\,\alpha(4\alpha(2\alpha+1)A + B)}{2A^2(\alpha+1)(2\alpha+1)^2(2+d)} + \cdots$$

The evolution towards the origin is started at a relative large value $\phi = \phi_{\rm max}$ using a large order asymptotic expansion (several terms).

The symmetry conditions at the origin ($v'(0) = 0$, $h(0) = 0$) are used to fix the free parameters A and B.

This time let us analyse the case d=3 for X_f in the range $0.001 < X_f < 3$

For this analysis we stop to refine the A, B values when we reach $v_1 = v'(0)\,,\; h_0 = h(0) \simeq 10^{-8}$

Analysing the two dimensional vector field $(v_1, h_0)(A, B)$ generated by the ODE's in most cases in very few steps we converge to the solution within the prescribed error.

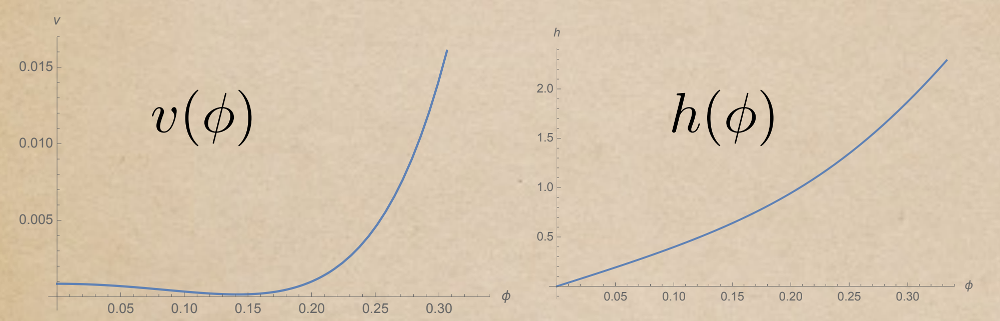
Therefore by construction the corresponding global scaling solutions do exist in the LPA.

For larger values of X_f more accuracy in the numerical evolution is required since derivatives of the functions appears to grow.

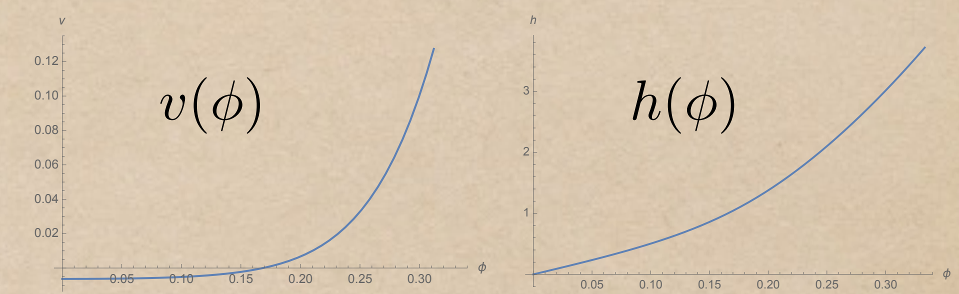
Numerical evolution from the asymptotic region

Some properties of the fully non trivial LPA scaling solutions in d=3:
if $X_f < 1.64$ the scalar is in the broken phase.

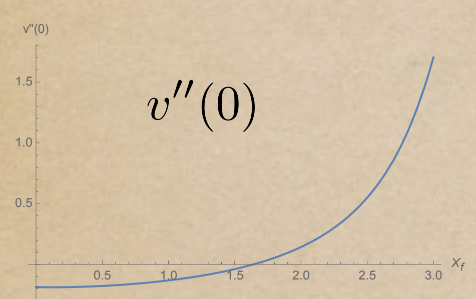
Illustrate with plots:

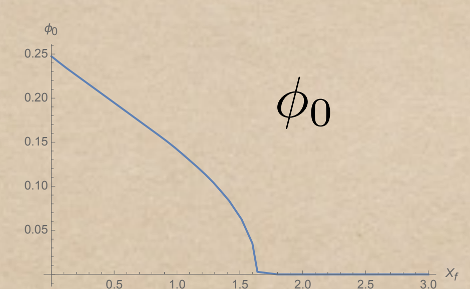


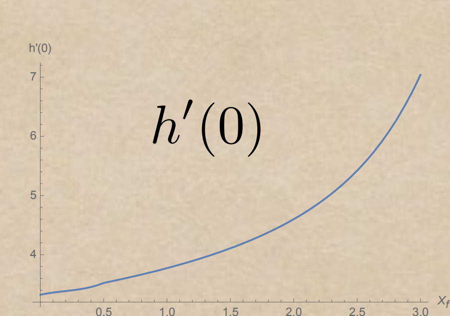
X_f =1

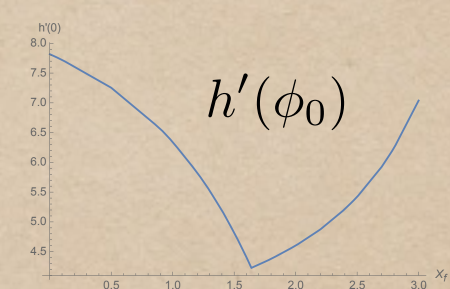


X_f =2



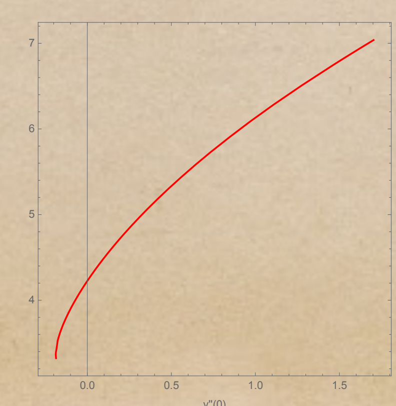






$0.001 < X_f < 3$: minimum of the scalar potential in ϕ_0 ($v'(\phi_0) = 0$)

Locus of the solutions in the plane
$(v''(0), h'(0))$ as function of X_f



Polynomial truncations

For d=3 they may provide a reasonable approximation. We have considered expansions both around the origin and a non trivial vacuum of the scalar field. We have worked with

$$u\,(\rho) = v(\phi) \quad , \quad h_2(\rho) = h^2(\phi) \quad , \quad \rho = \frac{\phi^2}{2}$$

Expansions:

around the origin:

$$u(\rho) = \sum_{n=0}^{N_v} \frac{\lambda_k}{k!}\rho^k \qquad h_2(\rho) = \sum_{n=1}^{N_h} \frac{y_k}{k!}\rho^k$$

around a non trivial vacuum:

$$u(\rho) = \lambda_0 + \sum_{n=2}^{N_v} \frac{\lambda_k}{k!}(\rho-\kappa)^k \qquad h_2(\rho) = \sum_{n=1}^{N_h} \frac{y_k}{k!}\left[(\rho-\kappa)^k - (-\kappa)^k\right]$$

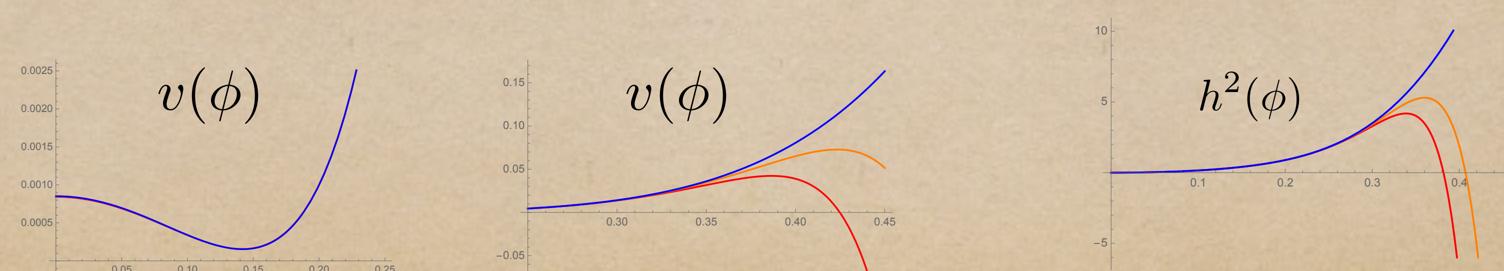
Polynomial analysis of the fully non trivial fixed point.

θ_i are minus the leading eigenvalues of the stability matrix

(N_v, N_h)	(4,3)	(5,4)	(6,5)	(8,7)	(9,8)
λ_1	-0.1209	-0.1315	-0.1339	-0.1315	-0.1309
λ_2	10.60	11.05	11.16	11.09	11.06
λ_3	293.2	339.6	351.0	342.7	340.1
y_1	26.84	28.38	28.76	28.53	28.44
y_2	986.6	1161	1206	1178	1167
θ_1	1.354	1.262	1.222	1.225	1.236
θ_2	-1.637	-1.047	-0.7554	-0.5809	-0.5893
θ_3	-2.427	-2.063	-1.738	-1.424	-1.448

(N_v, N_h)	(5,4)	(6,5)	(7,6)	(8,7)	(9,8)
κ	0.01000	0.01013	0.01006	0.01006	0.01007
λ_2	15.58	15.17	15.30	15.28	15.28
λ_3	521.8	498.9	503.0	502.0	502.3
y_1	44.59	43.00	43.51	43.44	43.43
y_2	1924	1818	1842	1837	1837
θ_1	1.158	1.125	1.138	1.138	1.137
θ_2	-0.7134	-0.6661	-0.6262	-0.6407	-0.6424
θ_3	-1.841	-1.530	-1.530	-1.543	-1.558

Comparison with exact solution



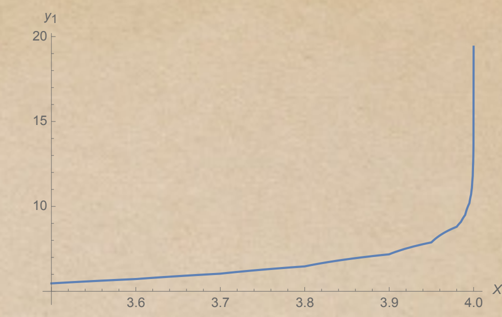
Polynomial analysis

In d=3 we find that the family of non trivial fixed point solutions encounters a singularity at $X_f = 4$. In particular we find that

$$X_f \to 4$$

$$y_1 \sim \frac{1}{4 - X_f}$$

$$\theta_1 \to 1$$

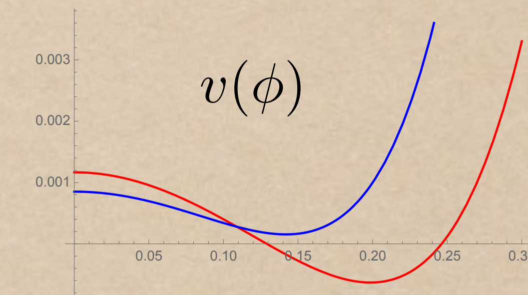


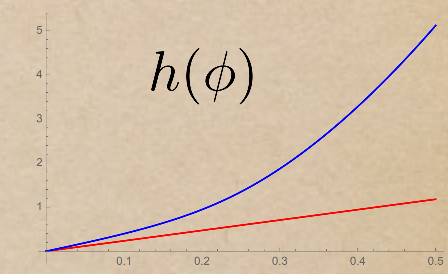
For $X_f > 4$ we have not been able to find a solution from the numerical analysis of the system of ODE's.

Standard Yukawa interaction
Nh=1 (d=3).

Leading critical exponent
almost 20% larger.

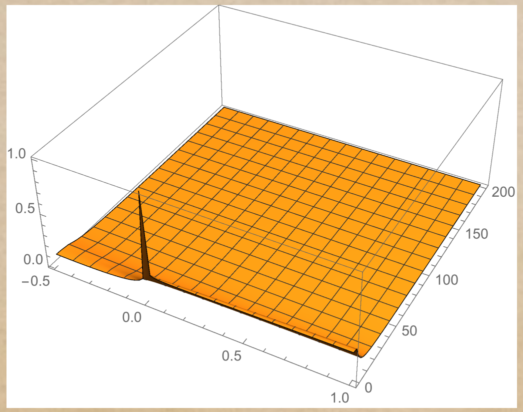
(N_v, N_h)	(4,1)	(5,1)	(6,1)	(8,1)	(9,1)
λ_1	-0.1602	-0.1742	-0.1765	-0.1720	-0.1716
λ_2	7.128	7.204	7.214	7.193	7.191
λ_3	121.9	134.7	136.7	132.7	132.4
y_1	11.35	11.06	11.01	11.11	11.11
θ_1	1.492	1.436	1.417	1.431	1.435
θ_2	-1.541	-1.184	-0.9628	-0.8584	-0.9049
θ_3	-12.27	-8.744	-6.395	-3.847	-3.415





Case d=4: A polynomial analysis with a standard Yukawa term (Gies, Scherer 2010) was showing that for $X_f \ll 1$ there could be a non trivial fixed point solution.

In our approach, with a much larger truncation, we find that such a possibility is excluded in this truncation. We find only the gaussian fixed point. To match their analysis we show here the case $X_f = 0.4$



Similar plots for d>4
only gaussian solution

Eigenperturbations of the ODE's

Perturbing the flow equations around a global scaling solution ($\epsilon \ll 1$) as

$$v(\phi) = v^*(\phi) + \epsilon \delta v(\phi) \quad , \quad h_2(\phi) = h_2^*(\phi) + \epsilon \delta h_2(\phi)$$

one can write for $\delta f^T = (\delta v, \delta h_2)$ an eigenvalue problem

$$\left(\hat{O} - \lambda\right) \delta f = 0$$

In particular

$$0 = (\lambda - d)\delta v + \frac{1}{2}(d-2)\phi\, \delta v' + C_d \left(\frac{X_f}{(1+h_2)^2} \delta h_2 - \frac{1}{(1+v'')^2} \delta v'' \right)$$

$$0 = \quad (\lambda - 2)\delta h_2 + \left(\frac{d}{2} - 1\right) \phi\, \delta h_2' + C_d \Bigg[\delta h_2 \, (h_2')^2 \left(- \frac{2}{(1+h_2)^3 (v''+1)} - \frac{(3h_2^2 + 2h_2 + 1)}{2h_2^2 (1+h_2)^2 (1+v'')^2} \right) +$$
$$\delta h_2' \, h_2' \left(\frac{2}{(1+h_2)^2 (1+v'')} + \frac{(3h_2+1)}{h_2 (1+h_2)(1+v'')^2} \right) - \frac{\delta h_2''}{(1+v'')^2}$$
$$+ \delta v'' \frac{\left(2h_2 (h_2+1)^2 h_2'' - (h_2')^2 \left(h_2 (v''+5) + 3h_2^2 + 1\right)\right)}{h_2 (1+h_2)^2 (1+v'')^3} \Bigg]$$

For d=3 the perturbations have the following asymptotic behaviour after rescaling by a global factor

$$\delta v_{\text{Asympt}} = \phi^{6-2\lambda} + \phi^{-2\lambda-4} \frac{\left(450 A^2 \beta X_f + B^2 \left(-2\lambda^2 + 11\lambda - 15\right)\right)}{13500 \pi^2 A^2 B^2} + \mathcal{O}\left(\phi^{-8-2\lambda}\right)$$

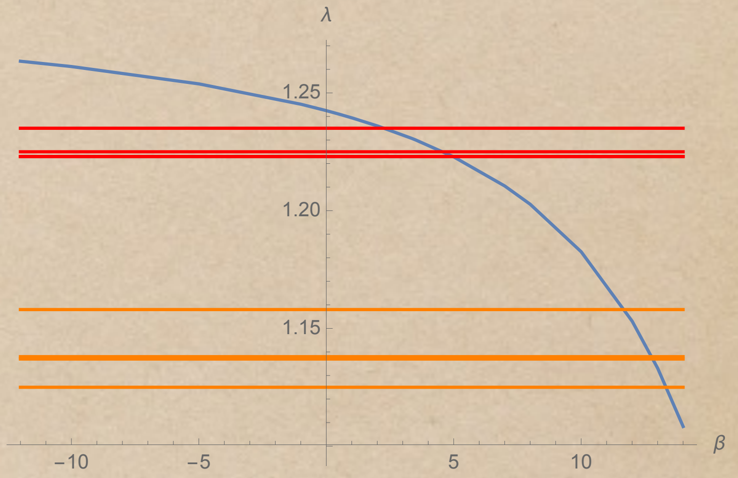
$$\delta h_{2,\text{Asympt}} = \beta \phi^{4-2\lambda} - \phi^{-2\lambda-6} \left(\frac{\left(2\lambda^2 - 11\lambda + 15\right)(20A + B)}{16875 \pi^2 A^3} + \frac{\beta \left(240 A \lambda + B \left(2\lambda^2 + 5\lambda - 6\right)\right)}{13500 \pi^2 A^2 B} \right) + \mathcal{O}\left(\phi^{-10-2\lambda}\right)$$

Numerical analysis of eigenperturbations (incomplete)

Imposing the asymptotic boundary conditions as well as the symmetry conditions at the origin we perform the numerical evolution. The two symmetry conditions should constraint the arbitrary values for β and λ

We find that imposing the symmetry condition within the same accuracy used for the search of the fixed point solution only restrict the values of β and λ within a piece of curve.

For the leading eigenvalue the result of this analysis together with the ones of the two polynomial truncations is not accurate enough.
Need to repeat the analysis with a higher precision.



Large number of fermions.

In the limit $X_f \to \infty$, since the scalar potential scales with X_f , the fixed point equations simplifies to

$$0 = -dv + \frac{d-2}{2}\phi\, v' - \frac{X_f\, C_d}{1+h^2} \qquad\qquad 0 = -h + \frac{d-2}{2}\phi\, h'$$

Solution

$$v(\phi) = c_2\, \phi^{\alpha\, d} - \frac{C_d}{d}\, {}_2F_1\left(1, -\frac{d}{2}; 1-\frac{d}{2}; -c_1^2\, \phi^{2\alpha}\right) \qquad h(\phi) = c_1\, \phi^\alpha \qquad \alpha = \frac{2}{d-2}$$

d=4 :

$$v(\phi) = c_2\phi^4 - \frac{C_d}{4} + \frac{C_d}{2}c_1^2\phi^2 + \frac{C_d}{2}c_1^4\phi^4 \log\left(\frac{\phi^2}{1+c_1^2\phi^2}\right) \qquad h(\phi) = c_1\, \phi$$

Eigenperturbations

$$0 = -(d+\lambda)\delta\mathrm{v} + \frac{1}{2}(d-2)\phi\delta\mathrm{v}' + \frac{2c_1 C_d X_f \phi^{\frac{2}{d-2}}}{\left(c_1^2\phi^{\frac{4}{d-2}}+1\right)^2}\delta\mathrm{h} \qquad 0 = -(\lambda+1)\delta\mathrm{h} + \frac{1}{2}(d-2)\phi\delta\mathrm{h}'$$

$$\delta h(\phi) = \phi^{\frac{2(\lambda+1)}{d-2}} \qquad \delta v(\phi) = c_2\phi^{\frac{2(d+\lambda)}{d-2}} - \frac{c_1^2 C_d X_f \phi^{\frac{2(d+\lambda)}{d-2}-2}\left(\frac{d-2}{c_1^2\phi^{\frac{4}{d-2}}+1} - d\, {}_2F_1\left(1, 1-\frac{d}{2}; 2-\frac{d}{2}; -\phi^{\frac{4}{d-2}}c_1^2\right)\right)}{d-2}$$

d=4: To have δh well behaved in the origin $\qquad \lambda = -1, 0, \cdots$

In this limit probably multifermions interactions are important.

Conclusions

- We have considered a Yukawa system in the LPA and shown the existence of the scaling solution beyond a polynomial truncations.
- The problem can be analyzed as a function of two parameters d and X_f When d approaches 2 a growing number of scaling solutions are found which should correspond to more general multi critical theories.
- We find that the analysis with a polynomial truncation can lead sometimes to wrong results, i.e. for the d=4 case.
- Truncation with multifermion interactions should give some quantitative corrections but the pattern found should not change for d>2.
- Also a general analysis in the next order in the derivative expansion should be probably carried on.
- More complicated QFT models which gets closer to SM should probably be analyzed taking into account for Yukawa terms such kind of truncations.