Properties of Two Lattice Models with Long-Range Couplings

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Outline

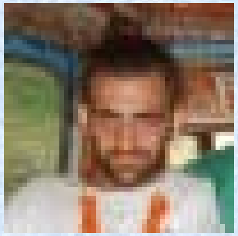
Two systems with long-range couplings:

- **lattice fermions → violation of the area law: <u>volume law</u> through formation of <u>Bell pairs</u>**

(if time permits)

- **Ising and O(N) models → studied by functional RG: <u>effective fractional dimension</u>**

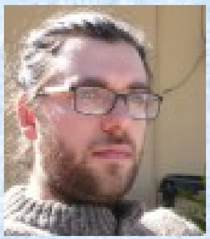
Tomorrow at the poster session see poster by Nicolo' Defenu →



Acknowledgements:

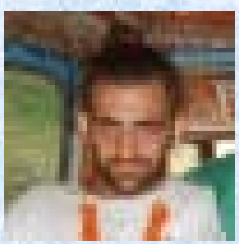
Volume law:

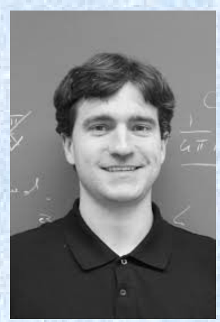
G. Gori, S. Paganelli, A. Sharma, P. Sodano



Natal (Brazil)

Trieste





Odense

N. Defenu and A. Codello

Ising model with long-range couplings:

Outline

Two systems with long-range couplings:

- **lattice fermions → violation of the area law: volume law through formation of Bell pairs**
- **Ising model → studied by functional RG: effective fractional dimension**

Entanglement in many-body systems

→ Nowadays a standard tool to characterize quantum states & systems

→ Characterization of quantum phase transitions

→ Possibility to detect novel quantum phases, without local order (topological phases)

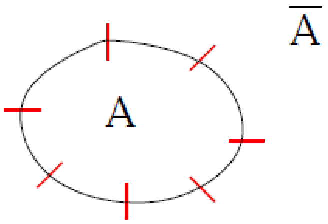
→ Characterization of the efficiency of algorithms such as DMRG

→ Thermal/classical states: extensivity of the entropy (no entanglement)

What about ground-state properties of a quantum system?

Entanglement Entropy (EE)

Consider a subsystem A:



$$\rho_A = Tr_{\bar{A}}\rho$$

Definition of EE $S_A = -Tr_A(\rho_A \ln \rho_A)$

Area law

Typically the EE grows as the boundary of the subsystem A:

$$\boxed{S_A \sim L^{d-1}}$$

For free fermions logarithmic corrections to the area law
[Wolff, PRL (2006); Gioev and Klich, PRL (2006)]

$$S_A \sim L^{d-1} \log L$$

Deviations from area law

Apart from logarithmic corrections, a deviation from area law is defined as

$$S_A \propto L^{\beta}$$

with

$$\beta > d - 1$$

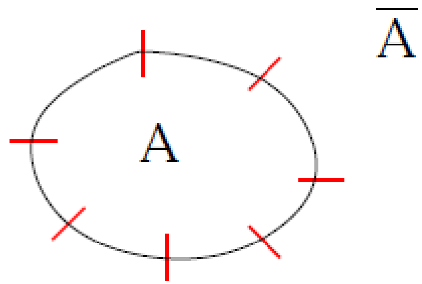
$$\text{in } d = 1 : \ \beta > 0$$

Volume law

$$\beta = d$$

$$\text{in } d = 1 : \ \beta = 1$$

A first attempt to have deviations from area law...



Since the number of links cut in the integration of the remaining part of the system is proportional to the boundary, then for inducing deviations from the area law...

...add non-local (eventually long-range) hoppings...

NOT WORKING!...BUT...

Entanglement Entropy for free fermions on a graph (I)

$$H = -\sum_{I,J} c_I^\dagger t_{IJ} c_J$$

$I, J = 1, \cdots, N_S$

filling $f = \breve{N}_T / N_S \ (0 \leq f \leq 1)$

EE on the subgraph A (having L sites, denoted by small letters):

$$S_A = -\sum_{\gamma=1}^{L} \left[(1 - C_\gamma) \ln (1 - C_\gamma) + C_\gamma \ln C_\gamma \right]$$

Entanglement Entropy for free fermions on a graph (II)

$$S_A = -\sum_{\gamma=1}^{L} \left[(1 - C_\gamma) \ln (1 - C_\gamma) + C_\gamma \ln C_\gamma \right]$$

C_γ eigenvalues of the correlation matrix [Peschel, JPA (2003)]

$$C_{ij} = \langle \Psi | c_i^\dagger c_j | \Psi \rangle = \sum_{\Gamma=1}^{N_T} \psi_\Gamma(i) \psi_\Gamma^*(j)$$

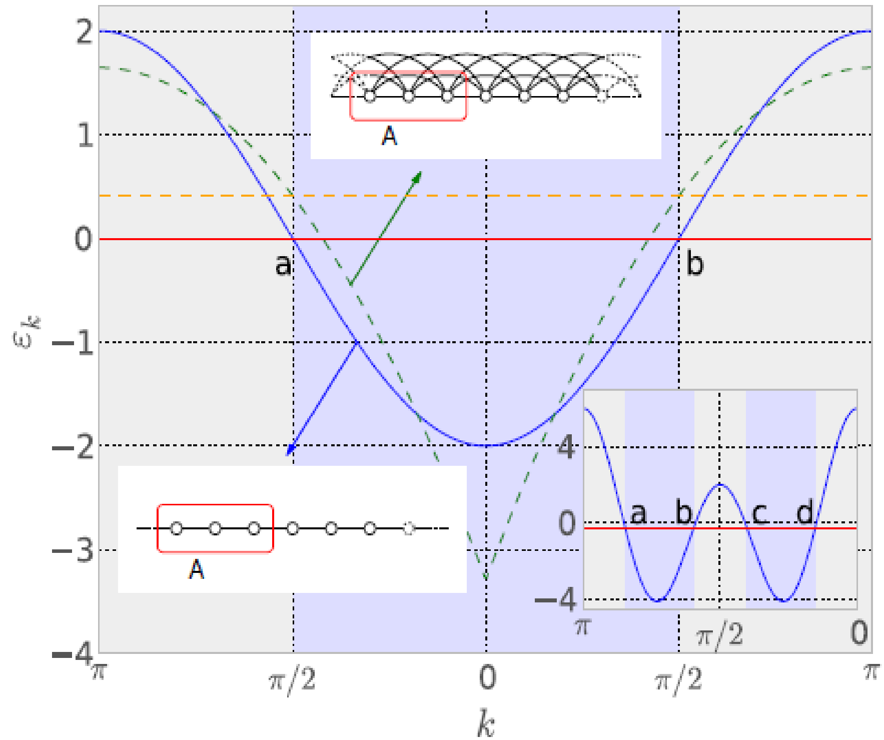
Of course, for translational invariant lattices it is

$$\psi_k(J) = \frac{1}{\sqrt{N_S}} e^{ikJ}$$

Power-law hoppings

$$t_{I,J} = \begin{cases} 0 & I = J, \\ \dfrac{t}{|I-J|_p^{\alpha}} & I \neq J, \end{cases}$$

$$|I-J|_p = \min(|I-J|, N_S - |I-J|)$$



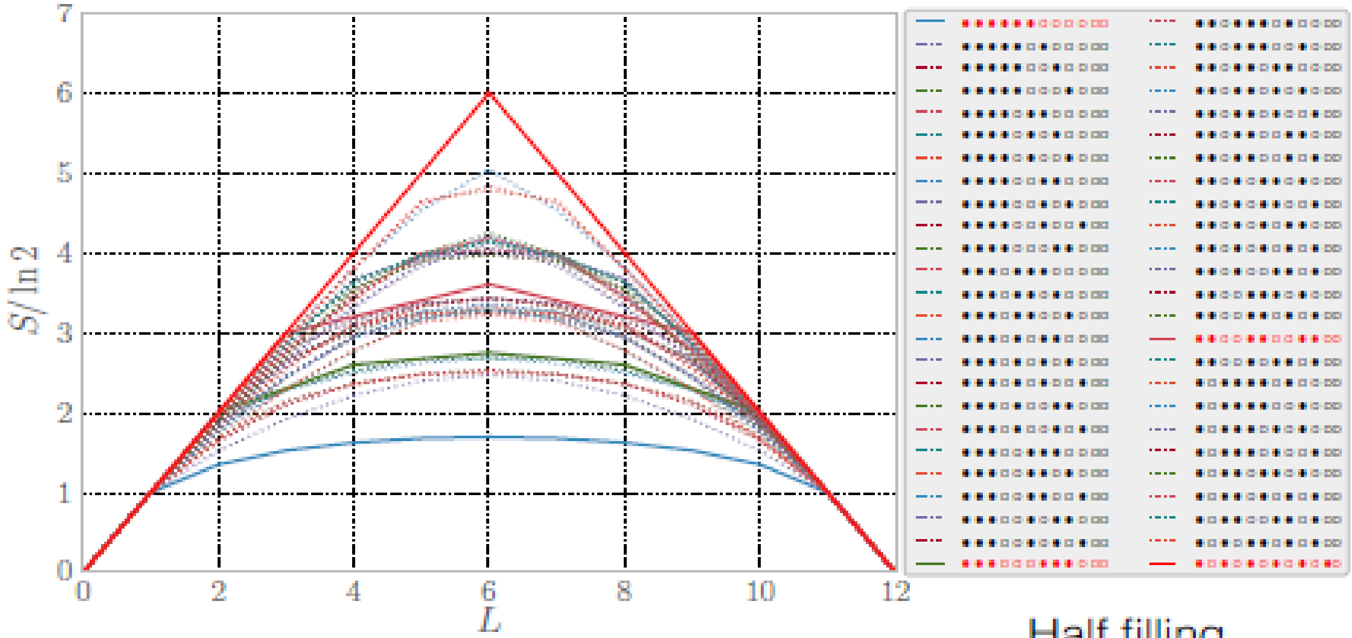
- Solid line: nearest neighbors
- dashed line $\alpha = 2$
- Inset: next-nearest neighbors model

Still area law for every positive α!

The reason is that the EE depends only on the topology of the Fermi surface

$$C_{ij} = \int_{k_a}^{k_b} \frac{dk}{2\pi} e^{ik(i-j)} \qquad C_{ij} = \int_{k_a}^{k_d} \frac{dk}{2\pi} e^{ik(i-j)} - \int_{k_b}^{k_c} \frac{dk}{2\pi} e^{ik(i-j)}$$

Let try all the possible Fermi surfaces:



Half filling
$N_S = 12$

- Maximal EE for alternating filling of the wave vectors
- Periodicity in k-space: piecewise linear behavior

A theorem on maximally EE states

→ Write the Fock space as $\mathcal{V} = \mathcal{A} \oplus \bar{\mathcal{A}}$

→ Construct two basis in the complementary systems

$$\alpha_j \in \mathcal{A},\ \bar{\alpha_j} \in \bar{\mathcal{A}}$$

$$\beta_1 = \frac{1}{\sqrt{2}}(\alpha_1 + \bar{\alpha}_1), \beta_2 = \frac{1}{\sqrt{2}}(\alpha_2 + \bar{\alpha}_2), \ldots, \beta_{\dim \mathcal{A}} = \frac{1}{\sqrt{2}}(\alpha_{\dim \mathcal{A}} + \bar{\alpha}_{\dim \mathcal{A}})$$

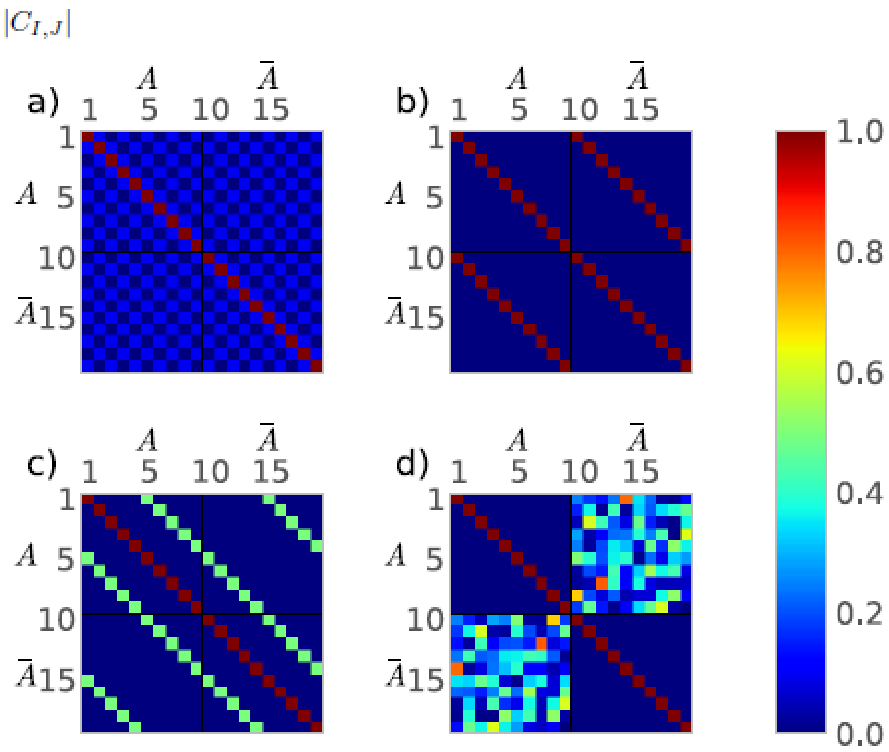
Single particle occupied states (not necessarily wave vectors) constructed in this way maximize the EE and obey the volume law.

→ maximize as well also all the Rényi entropies

→ Bell-paired states

The theorem is explicitly constructive in translational invariant systems.

In 1D translational invariant chains: states with maximal (volume law) EE have zig-zag occupation of the momenta k's → momenta k odd (or even) occupied, and the even (or odd) empty



A. ferromagnetic hopping
B. zigzag state
C. alternation of two filled and two empty
D. Bell-paired with random connection between the complementary subsystems

B → zig-zag state obeys the volume law (S is L ln2)

What is the Hamiltonian having the zig-zag states as ground state?

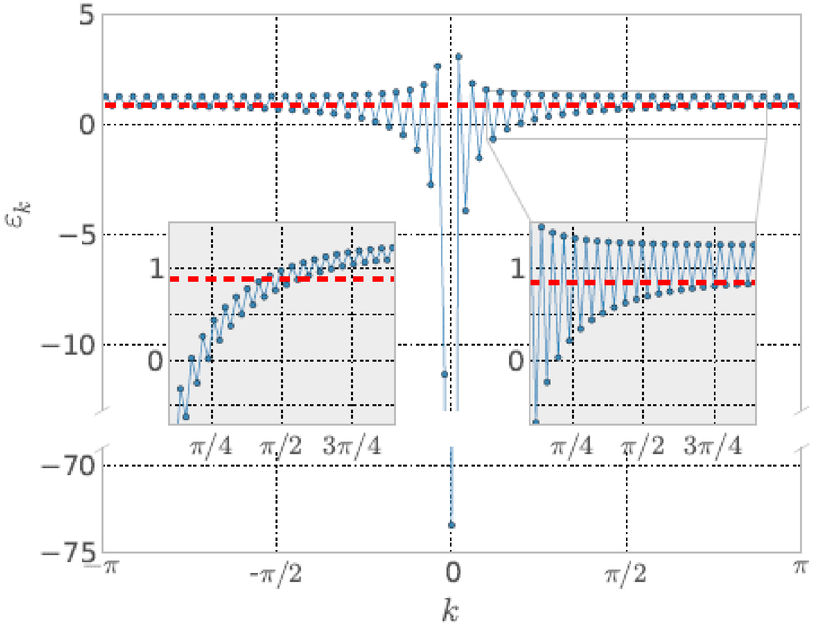
$$t_{I,J} = \begin{cases} -t & for\, |I-J|_p = \frac{N_S}{2} \\ 0 & otherwise, \end{cases}$$

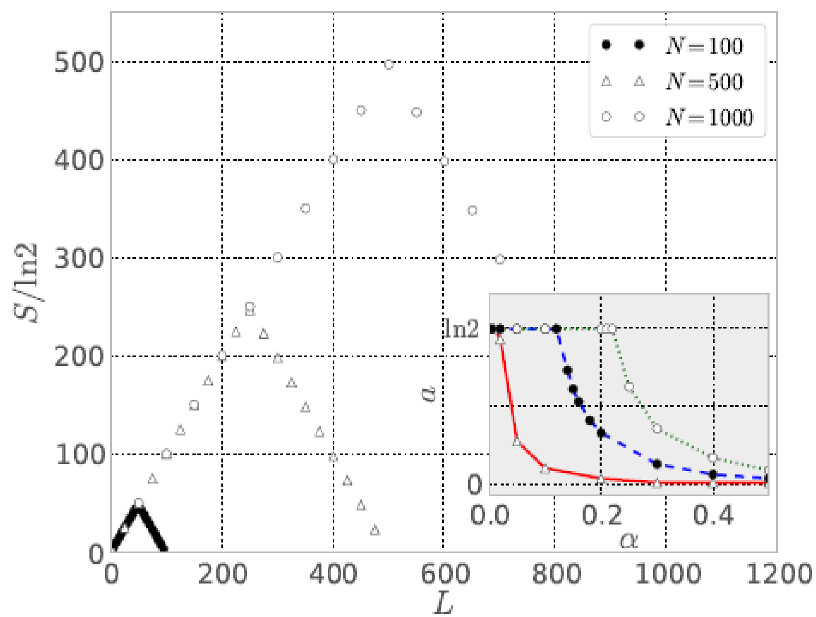
Other examples...

Long-range (α<1) + a magnetic field

$$t_{I,J} = \frac{t \cdot e^{i\phi d_{I,J}}}{|I-J|_p^\alpha}$$

$$\phi = \frac{2\pi}{N_S}\Phi$$





α=0.1
Φ=0.1

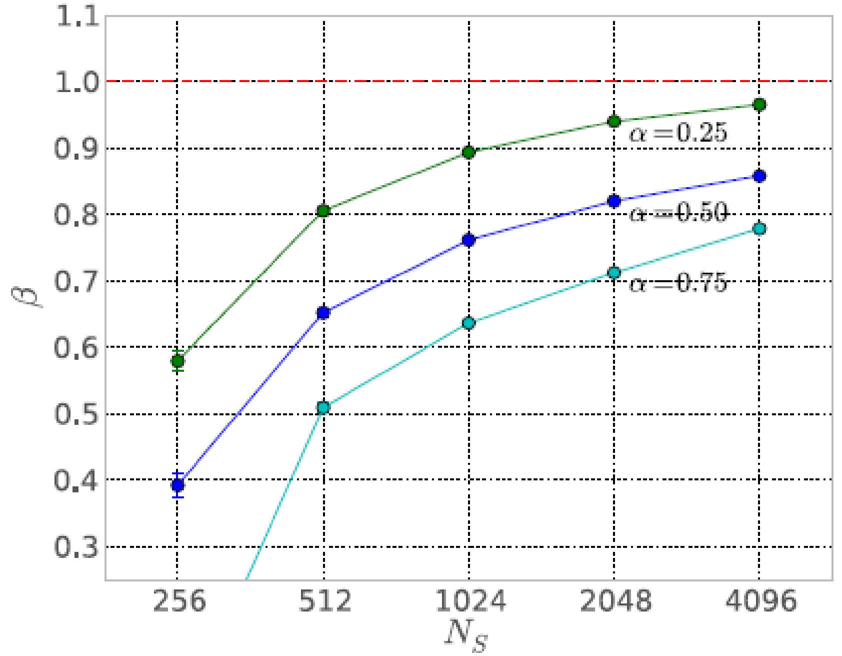
Inset: three values of Φ

$$S = aL + b$$

Long-range (α<1) + randomness

$$t_{I,J} = \frac{t \cdot \eta_{I,J}}{|I-J|_p^{\alpha}} \qquad \eta_{I,J} = \pm 1$$

Area law for α>1
Linear behaviour for α<<1



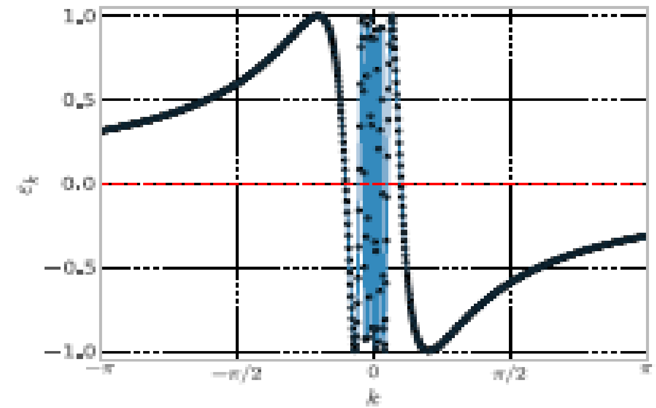
Relation with the box counting dimension of the Fermi surface:

Fermi surface with an accumulation point

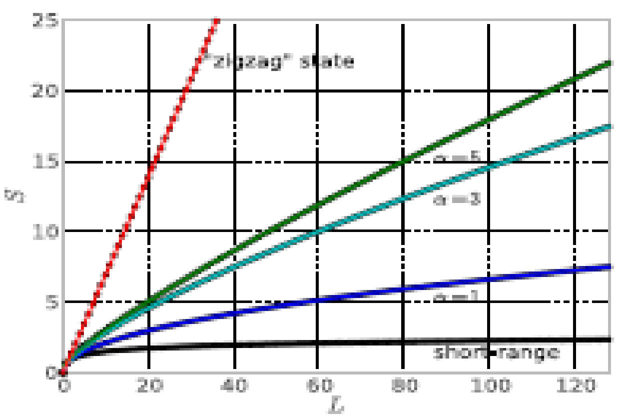
$$\varepsilon_k = -t \cdot \sin\left(\frac{1}{k^{\alpha}}\right),$$

Fermi surface:

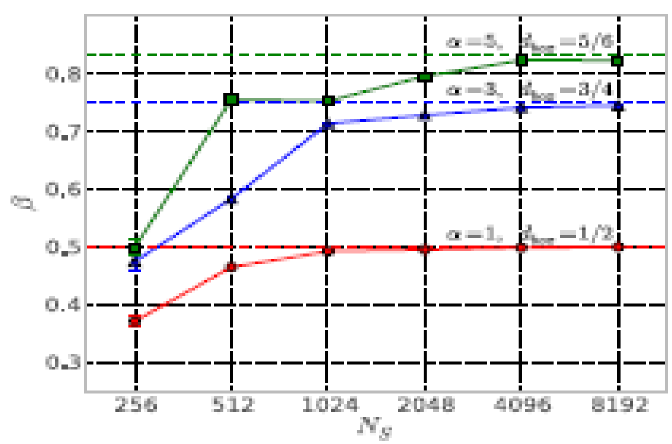
$$\{\pm\frac{1}{\pi}, \pm\frac{1}{\pi 2^{\alpha}} \pm \frac{1}{\pi 3^{\alpha}} \dots\}$$



Set with box counting dimension: $d_{box} = \dfrac{\alpha}{\alpha+1}$



L=1…128. Short range: S=a+b ln L



Fit function:

$$S = a + bL^{\beta}$$

Conclusions

- **Free fermions on a lattice → violations of the area law related to the structure of the Fermi surface**
- **volume law through formation of Bell pairs**
- **what matters is the topology of the Fermi surface**
- **long-range alone is not enough, but when combined with a magnetic flux or randomness gives volume law**

Outline

Two systems with long-range couplings:

- **lattice fermions → violation of the area law: volume law through formation of Bell pairs**
- **Ising model → studied by functional RG: effective fractional dimension**

Long-range O(N) model

$$H = -\frac{J}{2}\sum_{i \neq j} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{|i-j|^{d+\sigma}}$$

→ $\sigma \leq 0$ diverging energy – Kac rescaling

[Campa, Dauxois, Ruffo, Phys. Rep. (2009)]

→ $\sigma \leq d/2$ mean-field

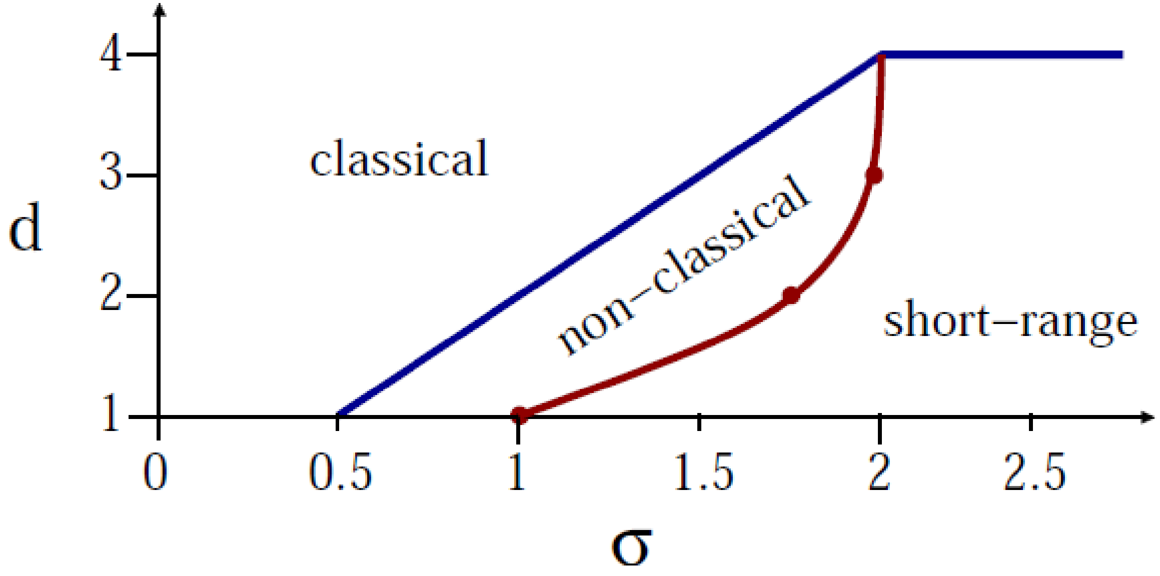
→ $\sigma > \sigma^*$ short-range critical exponents

→ $d/2 < \sigma \leq \sigma^*$ peculiar critical exponents

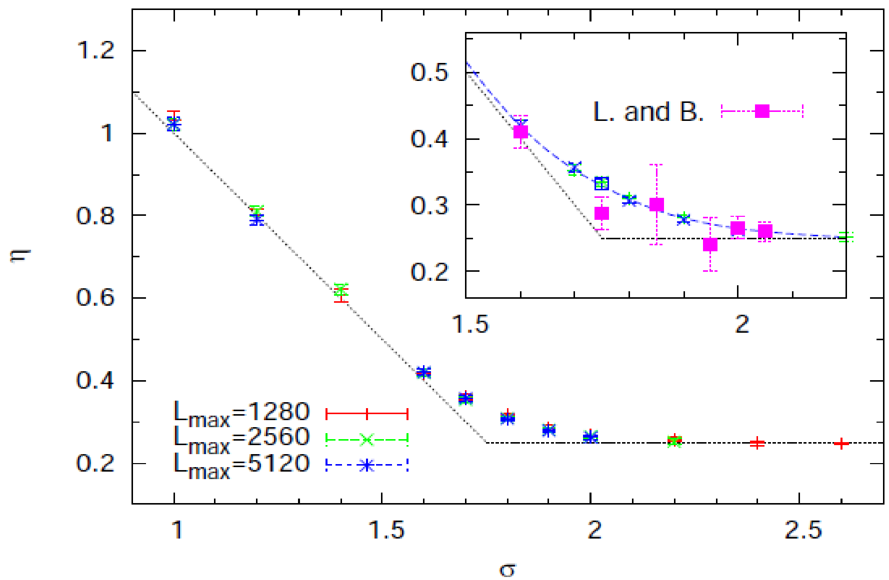
Early days Renormalization Group Prediction:

Sak, Phys. Rev. B (1973)

$$\sigma^* = 2 - \eta_{SR}$$



Recent Monte Carlo seems to question Sak's results for the Ising model (N=1)



$$\sigma^* = 2$$

M. Picco, arXiv (2012)

However, more recent Monte Carlo results do not seem to quite agree with Picco's results [M.C. Angelini, G. Parisi, and F. Ricci-Tersinghi, arXiv (2014)]

Our results (I)

$$\Gamma_k[\rho] = \int d^dx \left\{ Z_k \partial_\mu^{\frac{\sigma}{2}} \phi_i \partial_\mu^{\frac{\sigma}{2}} \phi_i + U_k(\rho) \right\}$$

→ in LPA one finds that the critical exponents of the Long-Range O(N) models are the same of the corresponding Short-Range O(N) models in an effective fraction dimension

$$D_{eff} = \frac{2d}{\sigma}$$

→ since LPA is exact is valid for N infinite, this result hold for the spherical model, retrieving an old result by Joyce (1972): our result extend it for all universality classes (tricritical, etc.)

Our results (II)

→ in LPA'

$$D'_{eff} = \frac{\left[2 - \eta_{SR}(D'_{eff})\right] d}{\sigma}$$

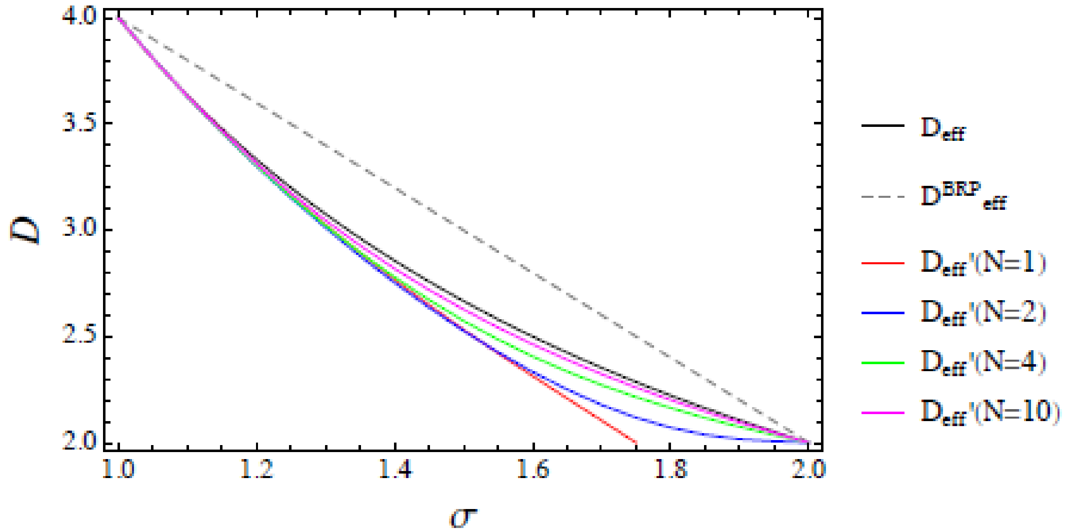
→ agreement with a result in M.C. Angelini, G. Parisi, and F. Ricci-Tersinghi, arXiv (2014)

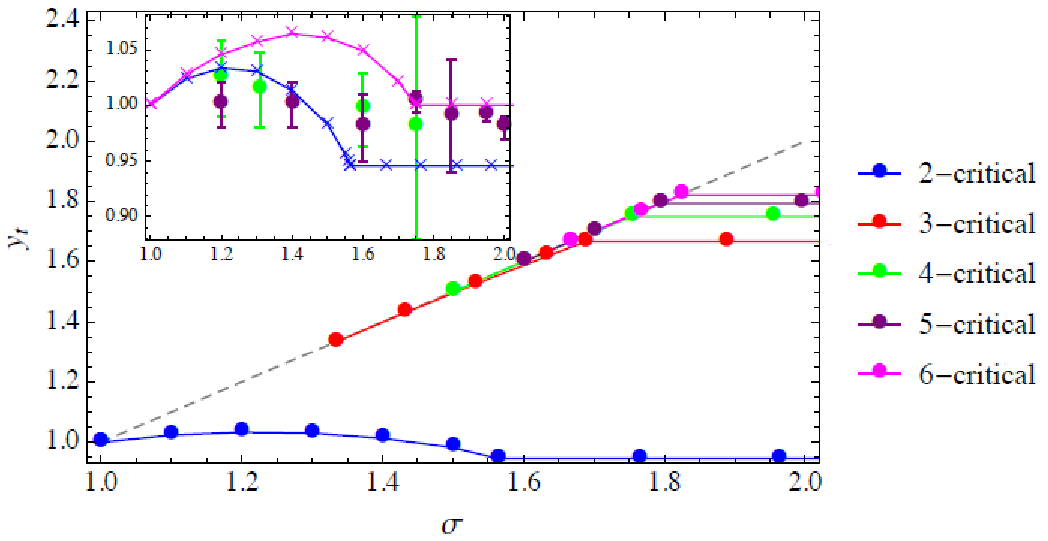
→ Sak's result is retrieved with the Litim cutoff

→ explicit expressions for the critical exponent ν and the universality classes

$$\nu_{LR}(d, \sigma) = \frac{2 - \eta_{SR}(D'_{eff})}{\sigma} \nu_{SR}(D'_{eff})$$

Our results (III)





Our results (IV)

→ **to treat on equal footing the terms q^2 and q^σ we devised an approximation we refered as LPA''**

$$\Gamma_k[\rho] = \int d^d x \left\{ Z_\sigma \partial_\mu^{\frac{\sigma}{2}} \phi_i \partial_\mu^{\frac{\sigma}{2}} \phi_i + Z_2 \partial_\mu \phi_i \partial_\mu \phi_i + U_k(\rho) \right\}$$

→ **use the regulator**

$$R_k(q) = (k^\sigma - q^\sigma)\theta(k^\sigma - q^\sigma) + (k^2 - q^2)\theta(k^2 - q^2)$$

→ **final result: Sak's result is yet retrieved and the Long-Range fixed point emerge at σ^***
(see details in the poster by Nicolo' Defenu)

Conclusions

- **Ising model & O(N) with long-range couplings → a functional RG study shows equivalence of critical exponents with the short-range model in an effective fractional dimension**

- **LPA result exact in the spherical model with long-range interactions → D_{eff} exact for large N**

- **Approximate effective dimension obtained also in LPA'**

- **In the approximation LPA'' the two terms q^2 and q^σ are treated on equal foot → Sak's result is retrieved**

Summary

Two systems with long-range couplings:

- **lattice fermions → violation of the area law:** **<u>volume law</u> through formation of <u>Bell pairs</u>; a fined-tuned long-range coupling or long-range couplings + magnetic field or randomness give volume law**

- **Ising model → studied by functional RG:** **<u>effective fractional dimension</u>**

Thank you!

A “pathological” example: the fully connected lattice (α=0)

$$H = -\frac{t}{N_S}\sum_{I \neq J} c_I^\dagger c_J$$

$$S_A \approx -L\left[(1-f)\ln(1-f) + f\ln(f)\right]$$

But the mutual information is vanishing:

$$I\left(A:\bar{A}\right) = S(A) + S\left(\bar{A}\right) - S\left(A,\bar{A}\right) = S_A + S_{\bar{A}} - S_T$$

For large N_s:

$$I \simeq 0$$

Fully connected case $\alpha = 0$

$$H = -\frac{t}{N_S}\sum_{I\neq J} c_I^\dagger c_J,$$

$$\left\langle c_I^\dagger c_J \right\rangle = \begin{cases} f & for \quad I = J \\ b & for \quad I \neq J \end{cases}$$

$N \to \infty$

- The GS is (N-1)-fold degenerate
- Fermi surface not well defined
- The EE scales linearly (but because of the degeneracy, is not an entanglement measurement)
- The Mutual Information could be used instead.
- No entanglement: reduction to a classical case

$$S_A \approx -L\left[(1-f)\ln(1-f) + f\ln(f)\right]$$

$$I\left(A:\bar{A}\right) = S(A) + S\left(\bar{A}\right) - S\left(A,\bar{A}\right) = S_A + S_{\bar{A}} - S_T$$

$N \to \infty \quad I \simeq 0$

Functional Renormalization Group results (FRG)

[with N. Defenu and A. Codello]

→ Local Potential Approximation (LPA): neglect anomalous dimension and the simplest dependence of the effective action on the field

→ LPA': retains the effect of the anomalous dimensions

Critical exponents and universality classes of short-range O(N) models well captured in LPA' also in effective fractional dimensions [A. Codello, PRL (2013)]