Anomalous scaling at non-thermal fixed points of Gross-Pitaevskii and KPZ turbulence

Thomas Gasenzer Steven Mathey Jan M. Pawlowski

ITP - Heidelberg

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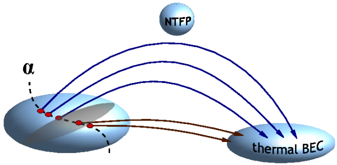
Non-thermal fixed points

Non-thermal fixed point are far from equilibrium quasi stationary states of matter.



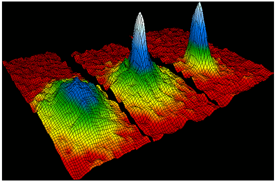
Scale invariance $\epsilon(k) \sim k^{-d+\eta}$

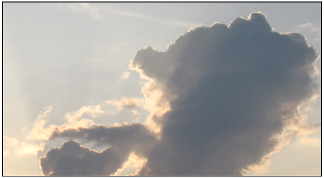
Depending on the initial conditions, the system may take an **algebraically long** time on the way to thermalisation.

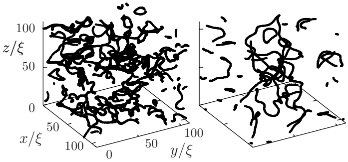
Plan

Ultra-cold Bose gases,
Gross-Pitaevskii Equation (GPE)

Interface dynamics,
Kardar-Parisi-Zhang (KPZ)







"Burn paper" by CrazzHky, used under CC BY / Cropped, http://crazzhky.deviantart.com/art/Burn-paper-288100073;
NIST/JILA/CU-Boulder; Nowak et al., arXiv:1206.3181v2

Driven-Dissipative GPE

Classical field equation for the average Bose wave-function $\phi(\mathbf{x}, t)$:

$$\mathrm{i}\partial_t \phi(\mathbf{x}, t) = \left[-\left(\frac{1}{2m} \qquad \right) \nabla^2 + \mu + g|\phi(\mathbf{x}, t)|^2 \right] \phi(\mathbf{x}, t)$$

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With complex parameters

$\mu = \mu_1 + i\mu_2$ — Single particle pump

$g = g_1 - ig_2$ — 2 particle losses

and stochastic driving

$$\langle \zeta(\mathbf{x}, t) \rangle = 0 \qquad \langle \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t') \rangle = \gamma\, \delta(t - t')\delta(\mathbf{x} - \mathbf{x}')$$

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Sieberer et al., PRL **110**, 195301 (2013); Sieberer et al., Phys. Rev. **B** 89, 134310 (2014);
Täuber et al., Phys. Rev. **X**, 021010 4 (2014)

Scaling in ultra-cold Bose gases

We focus on the kinetic energy density,

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$$\epsilon(k) \sim \quad k^{-d+\eta}$$

d	1	2	3
η_{num}	1	small	small

Nowak et al., Phys. Rev. B 84, 020506(R) (2011)

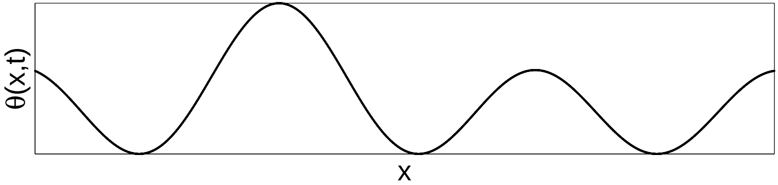
Kardar–Parisi–Zhang equation

A model for interface growth,

$$\partial_t \theta(\mathbf{x}, t) = \nu \nabla^2 \theta(\mathbf{x}, t) + \frac{\lambda}{2} \left[\boldsymbol{\nabla} \theta(\mathbf{x}, t)\right]^2 + \eta(\mathbf{x}, t)$$

with diffusion, perpendicular expansion and stochastic driving,

$$\langle \eta(\mathbf{x}, t) \rangle = 0 \quad \langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = D\, \delta(t - t') \delta(\mathbf{x} - \mathbf{x}')$$



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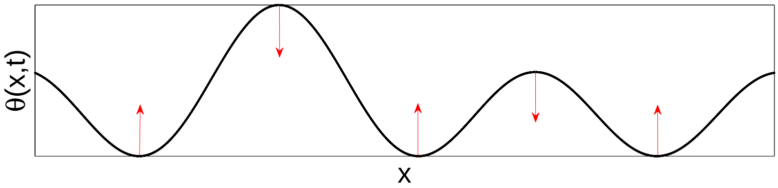

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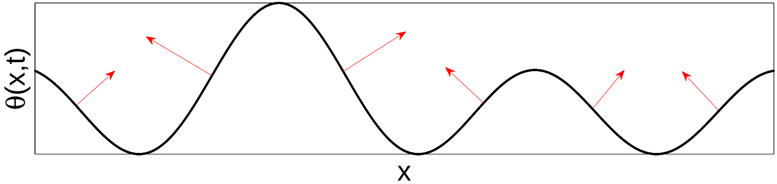

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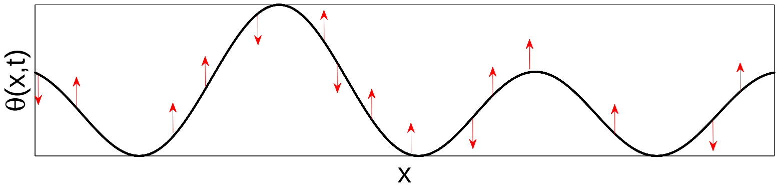

Scaling in interface growth

The stationary state has scaling correlation functions,

$$\langle \theta(t+\tau, \mathbf{x}+\mathbf{r})\theta(t, \mathbf{x})\rangle_c = r^{2\chi} g\left(\tau / r^z\right)$$

with exponents given by:

d	1	2	3	4
χ	$1/2$	0.379	0.300	?
$z = 2 - \chi$	$3/2$	1.6210	1.700	?

Kloss et al., Phys. Rev. E 86, 051124 (2012) and references therein

From DDGPE to KPZ

Density and phase decomposition $\phi(\mathbf{x},t) = \sqrt{n(\mathbf{x},t)}\,\mathrm{e}^{-i\theta(\mathbf{x},t)}$

$$\begin{aligned} \partial_t\theta - \frac{1}{2m}(\boldsymbol{\nabla}\theta)^2 - \nu\nabla^2\theta &= U, \\ \partial_t n - \frac{1}{m}\boldsymbol{\nabla}\cdot(n\boldsymbol{\nabla}\theta) &= S\ , \end{aligned}$$

with sources of phase and density fluctuations,

$$\begin{aligned} U &= U[\theta,n] + \frac{\mathrm{Re}(\zeta\mathrm{e}^{i\theta})}{\sqrt{n}} \\ S &= S[\theta,n] + 2\sqrt{n}\,\mathrm{Im}(\zeta\mathrm{e}^{i\theta}). \end{aligned}$$

Altman et al., arXiv:1311.0876v2 [cond-mat.stat-mech]

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$$\boxed{\partial_t \theta - \frac{1}{2m} (\boldsymbol{\nabla}\theta)^2 - \nu \nabla^2 \theta} = U, \qquad \text{KPZ equation}$$

$$\partial_t n - \frac{1}{m} \boldsymbol{\nabla} \cdot (n \boldsymbol{\nabla}\theta) = S\ ,$$

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Comparing scaling exponents

$$\epsilon_{\text{kin}} = \frac{1}{2m}\langle|\nabla\phi|^2\rangle \qquad \epsilon_{\text{kin}}(k) \sim k^{-d+\eta}$$

$$= \frac{n}{2m}\int_{\mathbf{k}} k^2\langle|\theta(\mathbf{k},t)|^2\rangle \qquad \epsilon_{\text{kin}}(k) \sim k^{z-d-\chi}$$

$$\rightarrow \boxed{\eta = z - \chi}$$

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GPE simulations

d	1	2	3
η_{num}	1	small	small

KPZ literature

d	1	2	3
η	1	1.242	1.400

Nowak et al., Phys. Rev. B 84, 020506(R) (2011)

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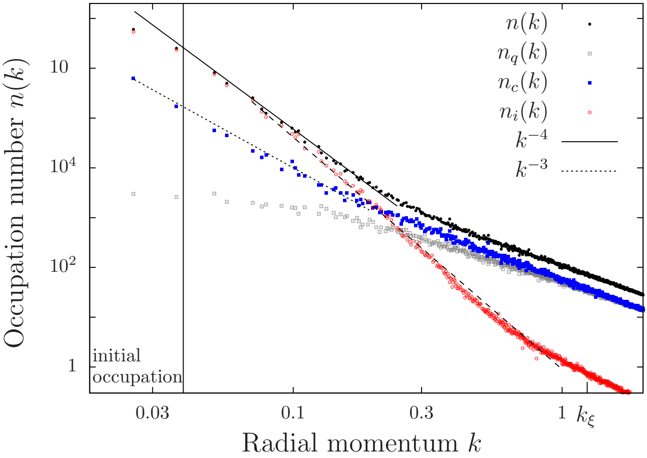
$d = 2$

$\epsilon(k) = k^2 n(k)$

$\nu = \mu_2 = 0$

$g_2 = \zeta = 0$



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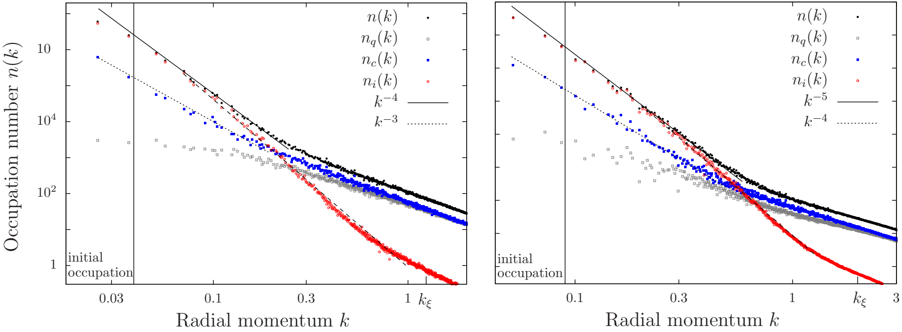

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Compressible excitations

$$\epsilon_c = \frac{1}{2m}\langle|\nabla\phi|^2\rangle \qquad \epsilon_c(k) \sim k^{-d+\eta+1}$$

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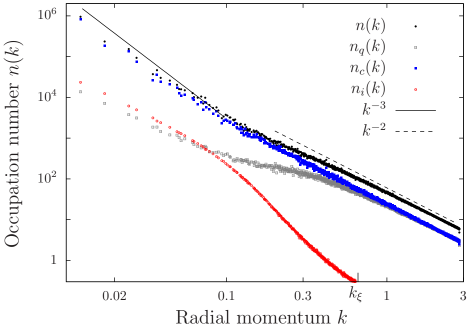
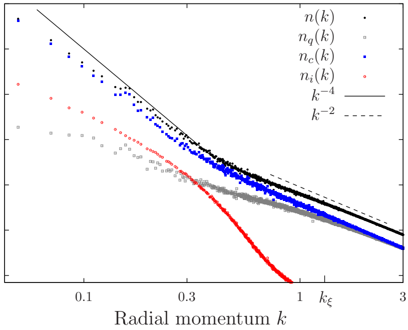


$d = 3$



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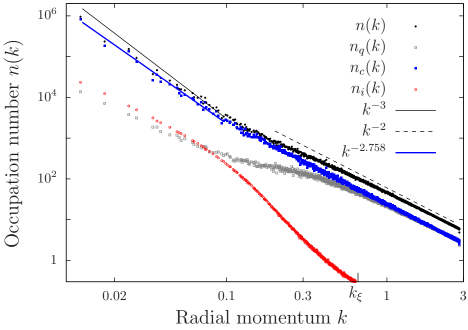

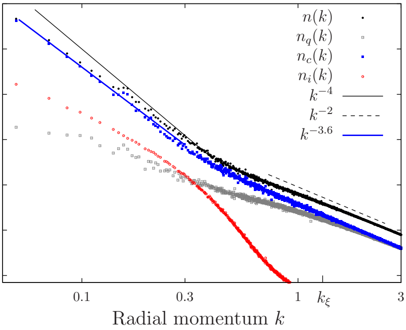

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Conclusions

- Interface dynamics described by KPZ equation does **not** capture **vortex** dynamics.
- It does captures **the rest**.
- We have made an estimation of **anomalous scaling exponents** of the ultra-cold Bose gas at a non-thermal fixed point.

Gasenzer, SM, Pawlowski, arXiv:1405.7652 [cond-mat.quant-gas]