Asymptotic Safety and Limit Cycles
in minisuperspace quantum gravity

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Based on arXiv:1205.4218 (and forthcoming), in collaboration with Daniel Litim

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Outline

1. Renormalization group for minisuperspace cosmology
2. RG flow, fixed points and limit cycle
3. Degenerate limit cycle at critical point
4. Flow for low *n* and large *n*
5. Beyond conformal reduction
6. Summary and outlook

1. Renormalization group for minisuperspace cosmology

We consider Euclidean GR restricted to spatially flat FRW metrics

$$\mathrm{d}s^2 = a^2(t)\left[\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right] \quad , \qquad S = \int dt\, \frac{3v}{8\pi G}\left[-a'(t)^2 + \frac{\Lambda}{3}a(t)^4\right]$$

and attempt to study the quantum theory through the Exact Renormalization Group.

Motivation: simplest theory that can be called “gravity”!

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As starting point we take the Conformally Reduced Einstein-Hilbert theory (CREH):

$$g_{\mu\nu} = \chi^2(x)\hat{g}_{\mu\nu}$$

$$\Gamma_k[\bar{f};\chi_B] = -\frac{3}{8\pi G_k}\int \mathrm{d}^4x\,\sqrt{\hat{g}}\left[-(\chi_B+\bar{f})\hat{\Box}(\chi_B+\bar{f}) + \frac{1}{6}\hat{R}(\chi_B+\bar{f})^2 - \frac{1}{3}\Lambda_k(\chi_B+\bar{f})^4\right]$$

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In our case we want a flat reference metric.

Dimensional reduction of fluctuations

We follow the derivation of the beta functions for G_k, Λ_k in the CREH theory, which are obtained by expanding the RG flow equation and matching terms

$$k\partial_k (G_k)^{-1} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathcal{F}(p^2, k)$$, simil. for Λ_k

Technical note: $\eta_{\mathrm{N}}^{(\mathrm{kin})}$!

Introduce a δ-function to suppress fluctuations in 4-n dimensions:

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$$\dot{\lambda}_k = (\eta-2)\lambda_k + \frac{g_k}{4\pi}\left(1-\frac{\eta}{n+2}\right)\frac{1}{1-2\lambda_k}$$

(Up to an overall scaling of g).

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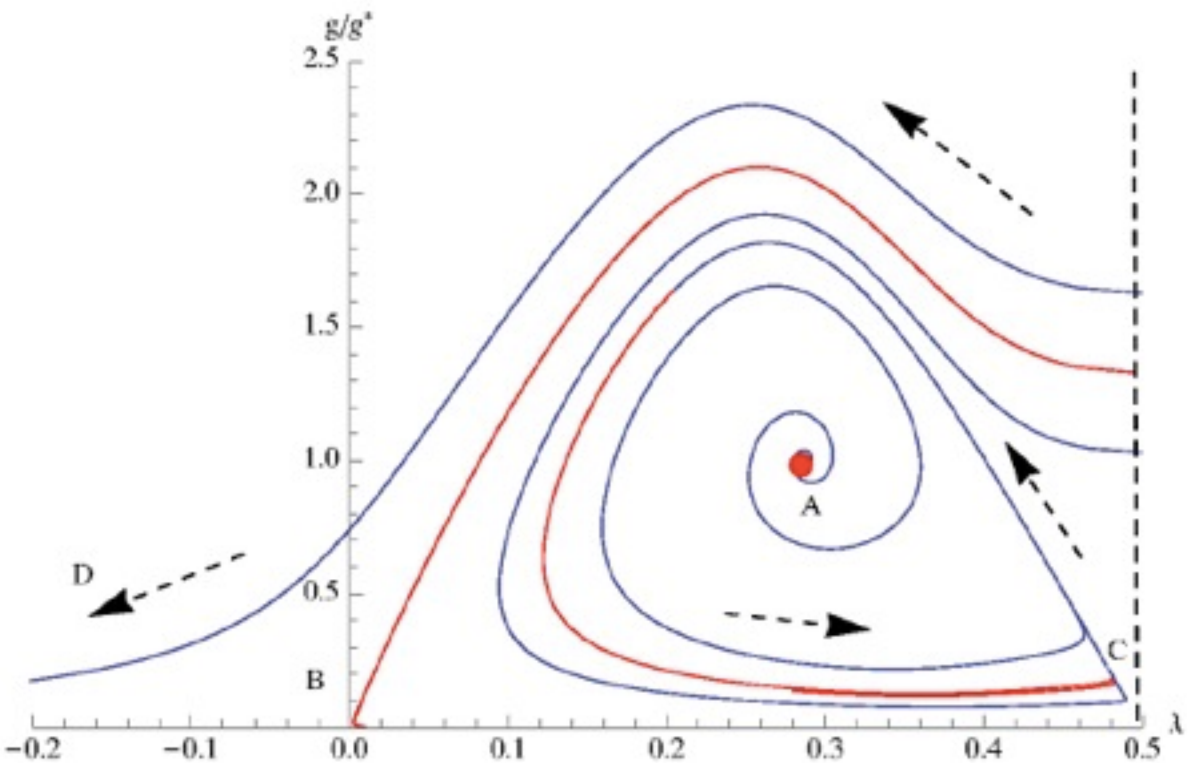
(Up to an overall scaling of g).

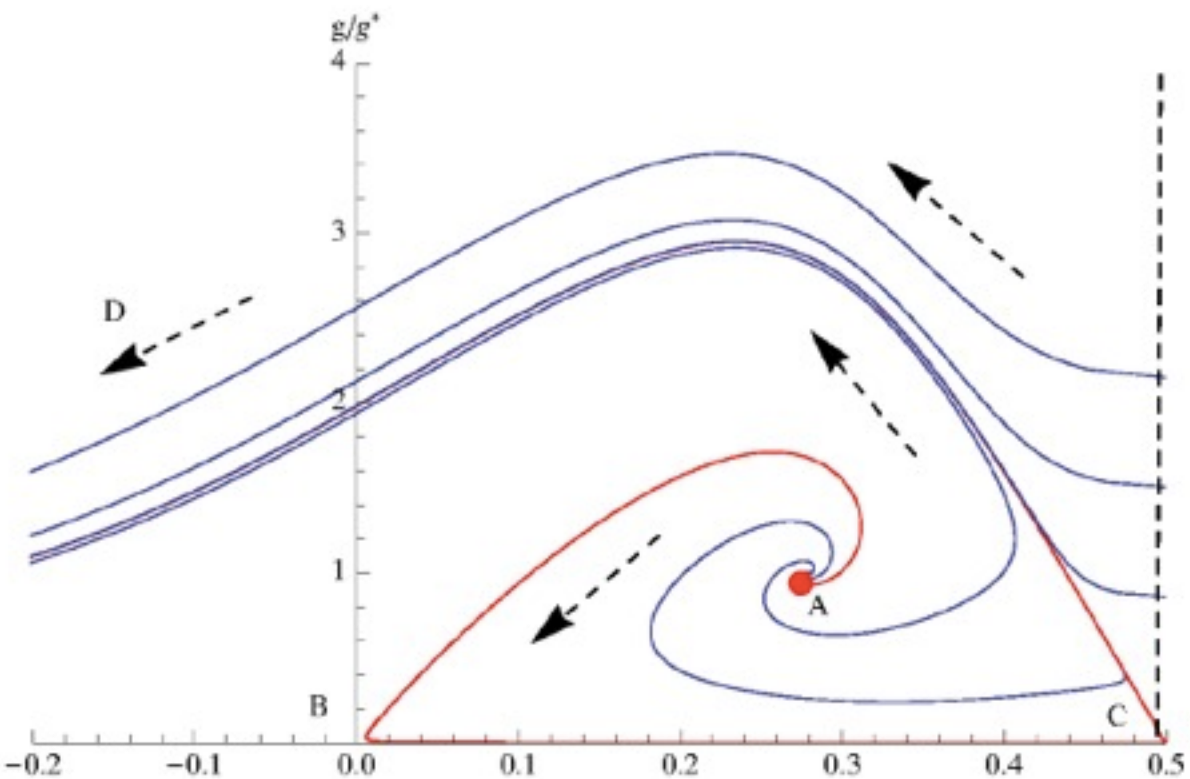
n = 1 → minisuperspace approximation

n = 4 → full CREH theory

2. RG flow, fixed points and limit cycle



n = 1 (minisuperspace)

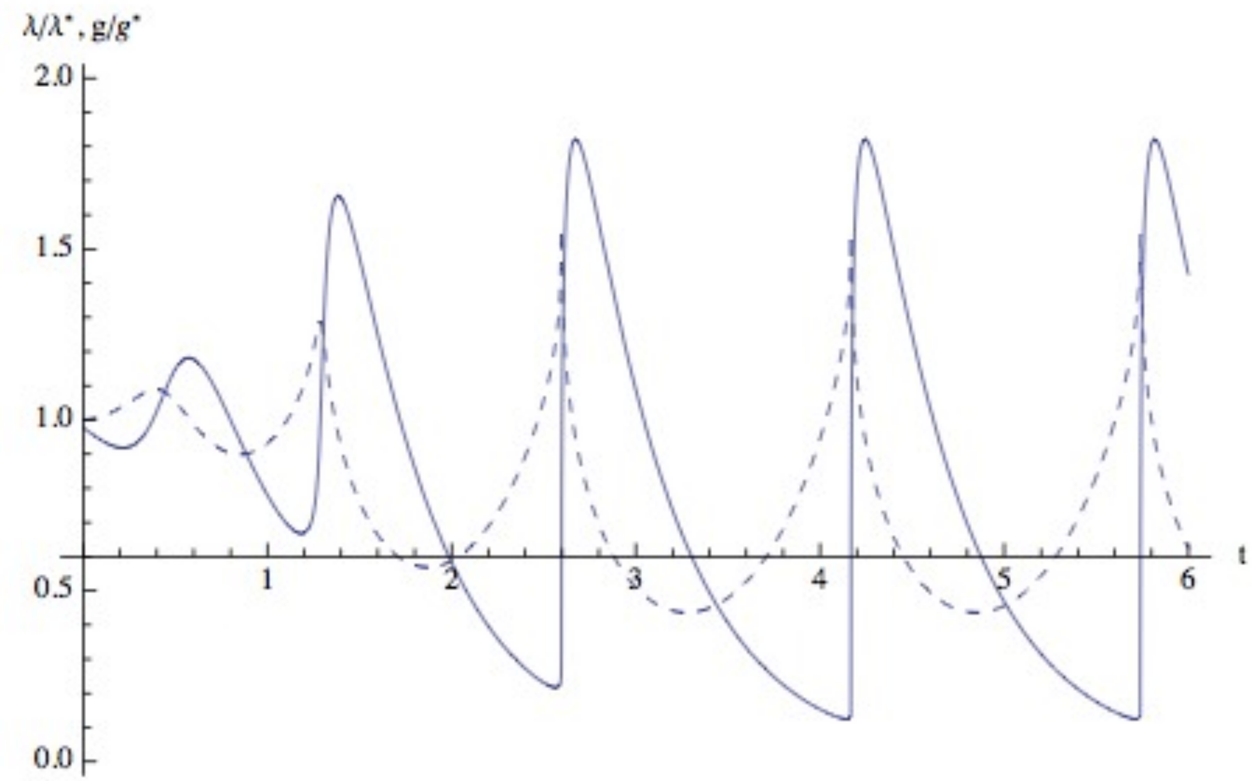


n = 4 (CREH theory)

In both cases:
- [A] UV-attractive non-Gaussian fixed point at positive (λ, g).
- [B] Gaussian fixed point.
- [C] degenerate fixed point at $(\lambda = 1/2,\ g = 0)$.
- [D] IR attractor at $(\lambda \to -\infty, g = 0)$.

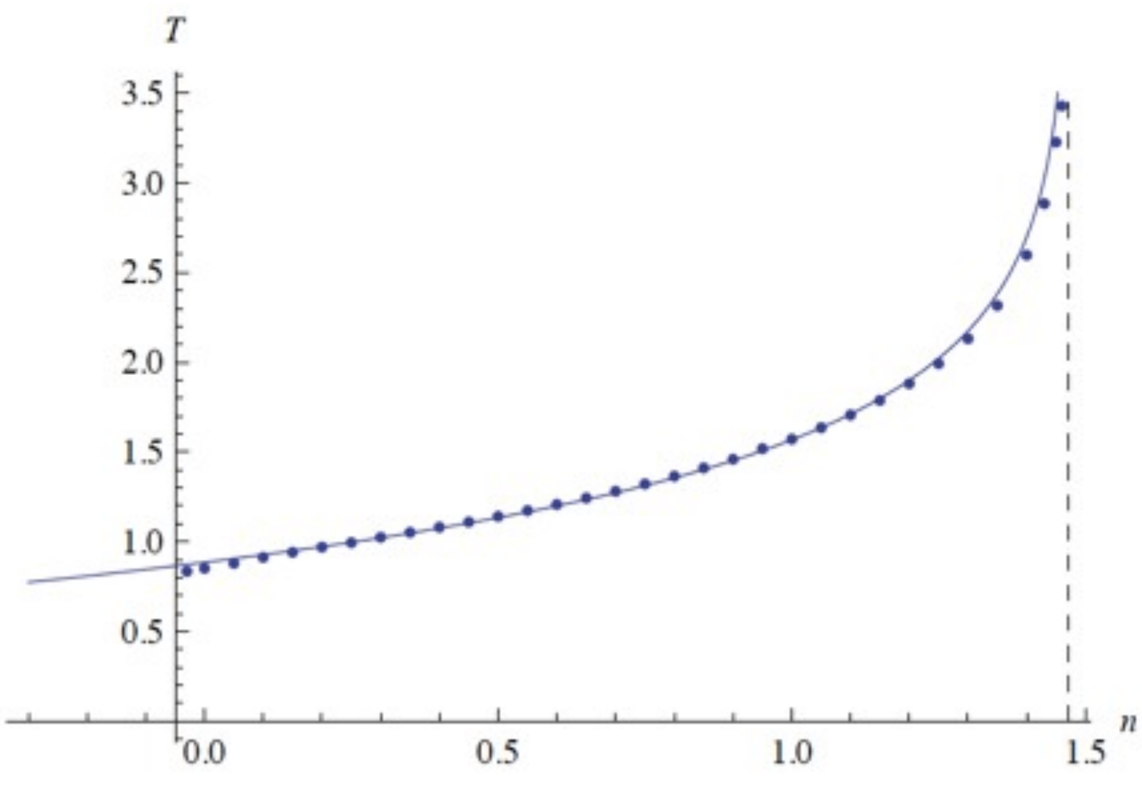
In minisuperspace case: also a limit cycle shielding the NGFP from the semiclassical regime. There are trajectories approaching the limit cycle from inside and outside, as well as others escaping towards $\lambda \to -\infty$.

Characterizing the limit cycle



The limit cycle at n = 1 has period T ≈ 1.57.

The cycle is not traversed uniformily. The flow makes a fast turn in the vicinity of the degenerate fixed point C.



We can also study the flow for continuous values of n. Increasing n, the period increases and is logarithmically divergent for $n \to n_{\text{crit}} \approx 1.4715$.

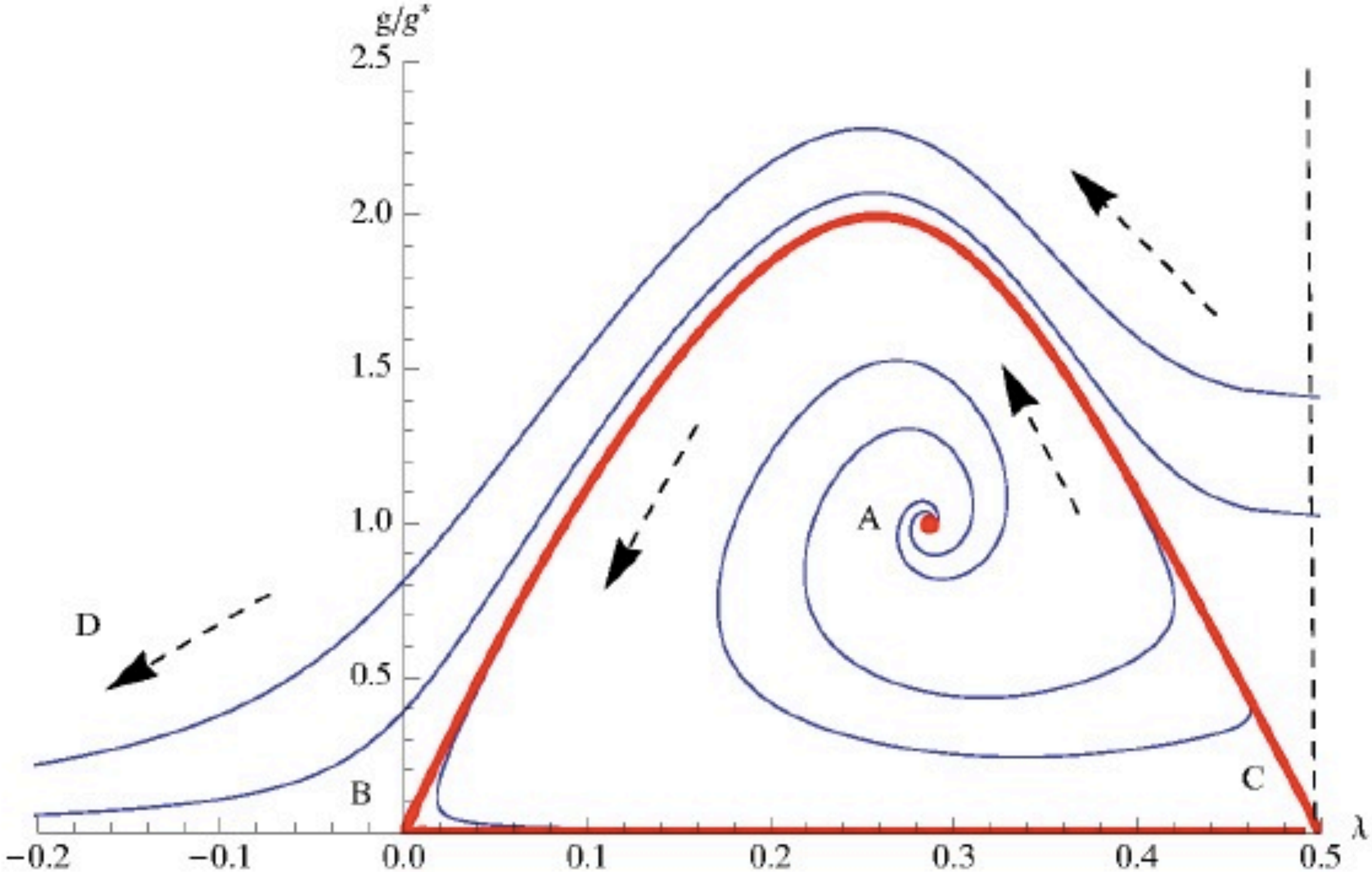
$$T_n = T_0 - b \ln\left(1 - \frac{n}{n_{\text{crit}}}\right)$$

$b \approx 0.57$.

3. Degenerate limit cycle at critical point

At $n = n_{crit}$, the limit cycle collides with the fixed points B and C and becomes degenerate:

Flow for $n = n_{crit} \approx 1.4715$.



All trajectories flowing into the IR from the NGFP approach asymptotically the degenerate limit cycle.
For $n > n_{crit}$, the limti cycle has vanished and the flow qualitatively resembles the full theory.

Implications for cosmological fine-tuning

For the Asymptotic Safety research program to deliver a viable cosmology, the physical values of G_k and Λ_k must be approximately constant (and small and positive) over the wide range of scales where they are measured.

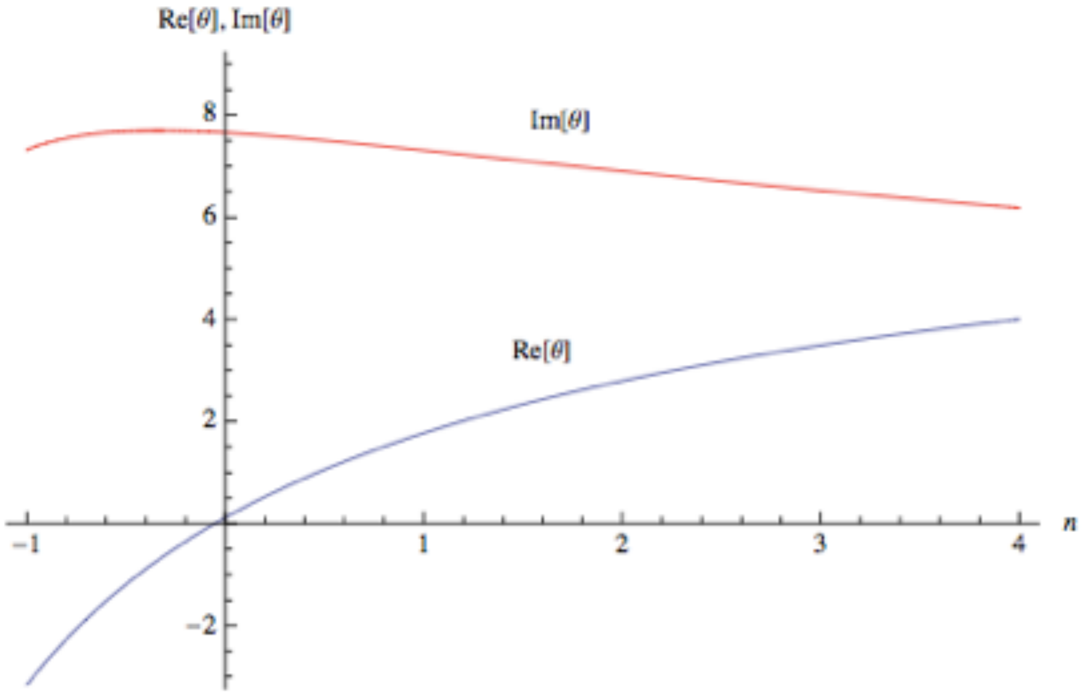
→ The RG flow trajectory realized in Nature must spend a large amount of RG "time" in the vicinity of the Gaussian fixed point, at $\lambda_k \gtrsim 0$ $g_k \gtrsim 0$.

For the usual EH and CREH truncations (and the minisuperspace too) this is not possible without fine-tuning the initial conditions of the RG flow.

For the critical value $n = n_{crit}$, all trajectories leaving the NGFP towards the IR achieve an extended semiclassical regime.

This suggest a new possible way in which the issue of the fine-tuning of the initial conditions for the flow might resolve itself.

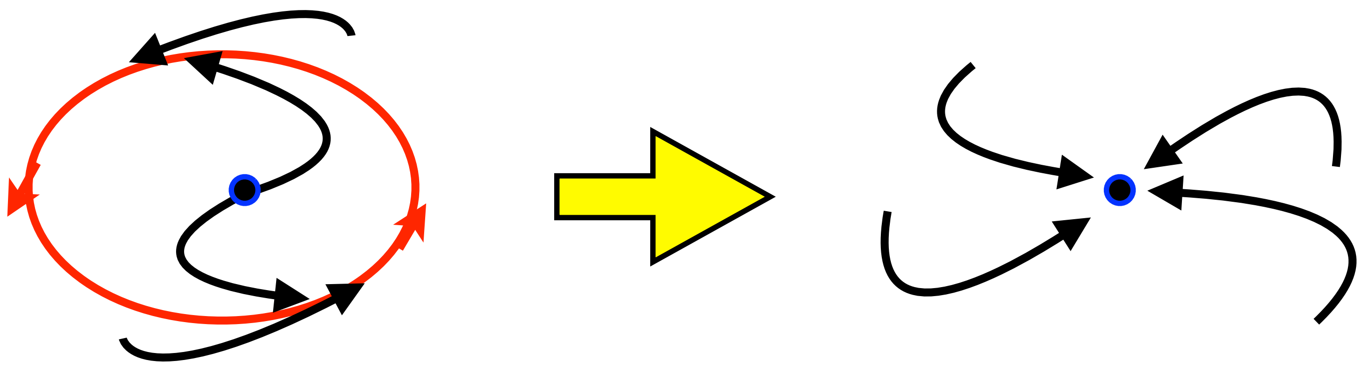
5. Flow for low n



For $n < 1$, the size of the limit cycle keeps decreasing as Re(θ^*) decreases and the NGFP becomes less strongly IR-repulsive.

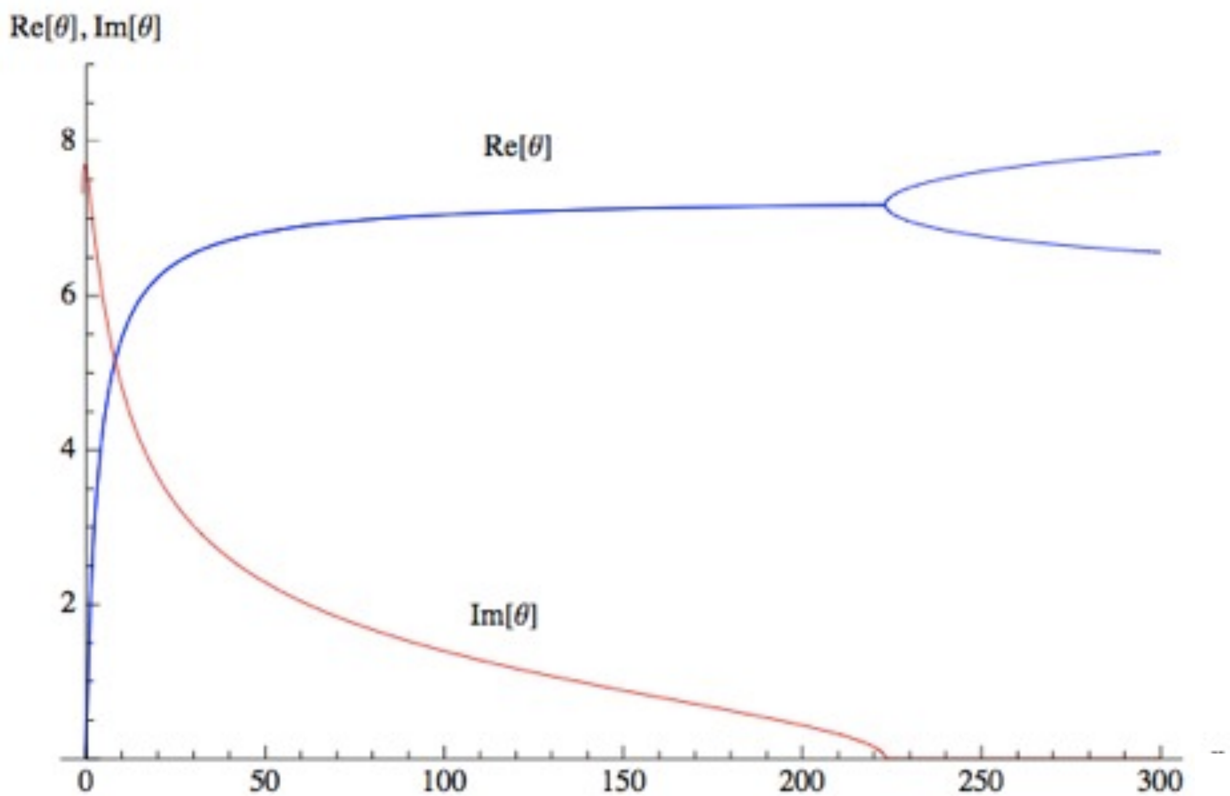
At $n \approx -0.05$, the limit cycle shrinks to a point and vanishes. Re(θ^*) becomes negative, and the NGFP becomes IR-attractive.

Hopf bifurcation pattern



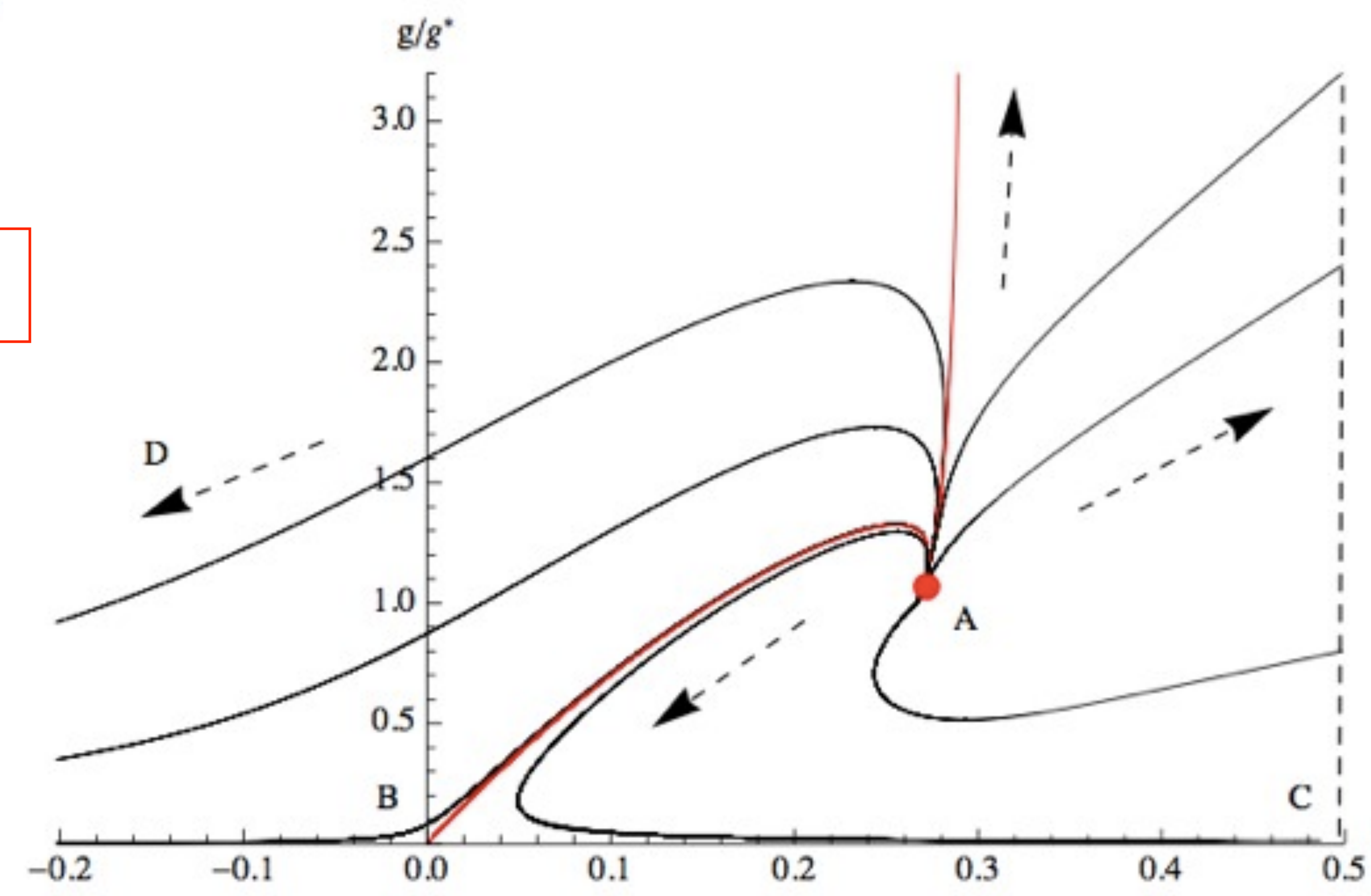
n does not have a physical interpretation in this regime.

Flow for large n



At $n+ \approx 223$, there is another bifurcation in which the critical exponents become real.

Flow for n ≫ n+.



6. Beyond conformal reduction

We can do a similar dimensional truncation of fluctuations on the traces that define the beta functions of the full Einstein-Hilbert theory.

In an approximation where the anomalous dimension is linear in g, we get:

$$\begin{aligned}
\partial_t g &= (2+\eta_N) g \,, \\
\eta_N &= -\frac{g}{3(n-2)} \frac{96-46n+12(5n-8)\lambda+(96-80n)\,\lambda^2}{(1-2\lambda)^2} \,, \\
\partial_t \lambda &= (\eta_N - 2)\lambda - 8g + 10g\left(1-\frac{\eta_N}{n+2}\right)\frac{1}{1-2\lambda} \,.
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Limit cycle still obtained, for $n < n_{crit} \approx 2.88$. This means the limit cycle is "close in theory space" to the full theory!

But equations are singular at $n = 2$. (Can we use $\eta_N^{(\mathrm{kin})}$?)

7. Summary and outlook

• The minisuperspace reduction of Einstein-Hilbert gravity presents a renormalization group limit cycle, absent when spatial fluctuations are preserved.

• The period of the limit cycle diverges at a critical value of the tuning parameter n, above which the theory resembles CREH. The critical exponents are real at large n. For low n, the limit cycle vanishes in a Hopf bifurcation.

• The theory at the critical point allows for an extended semiclassical regime with a small positive Λ with no need for fine-tuning the initial conditions.

• While this particular model with $n = n_{\text{crit}}$ is likely unphysical, it opens the door for a new way in which fine-tuning problems might resolve themselves in the Asymptotic Safety framework.

The n-tweaked theory is very close in "theory space" to the Einstein-Hilbert theory, as a "quick and dirty" calculation confirms, and we may hope the degenerate limit cycle may be a feature of the full theory that is lost in the standard approximation and can be found again with the "dimensional tweaking".

Thank You!