Classical and quantum aspects of unimodular gravity

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based on work together with Antonio Padilla, arXiv: 1409.3573
and work in progress to appear soon

ERG 2014, Leykada

The key points of this talk

- **The motivation:** Unimodular gravity and the cosmological constant problem
  $\rightarrow$ Does it really solve it?

- **Quantum unimodular gravity** and its UV structure under the light of the ERG

- **The (in)equivalence of quantum unimodular gravity** with quantum GR

Unimodular gravity and the cosmological constant problem

- The GR action: A Diff invariant and successful theory of gravity at solar and cosmological (?) scales

$$S = \int d^4x\sqrt{-g}\,\frac{R-2\Lambda}{16\pi G} + S_{\text{matter}}$$

- **Motivation:** The cosmological constant problem: Why is the observed value of Λ so small?
- **The idea** of unimodular gravity: Disentangle the coupling Λ from the classical dynamics of gravity [1]

$$\frac{\delta}{\delta g_{\mu\nu}}\sqrt{-g} = 0 \qquad \rightsquigarrow \quad \nabla_\mu \xi^\mu = 0 \ \text{(restricted symmetry:TDiff)}$$

- **The classical dynamics of unimodular gravity** are exactly the same with GR:

$$\text{Bianchi identities:} \quad R - 8\pi G T^\mu{}_\mu = \text{const.} \equiv 4\lambda_0$$

$$\text{Field equations:} \quad G_{\mu\nu} + \lambda_0 g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- So what changes then? Absolutely no change in physics!

Classical unimodular gravity = Classical GR

[1] The first one to introduce unimodular gravity was Einstein himself, but in a different context, A. Einstein, Annalen der Physik, vol. 354, 769?822 (1916)

Restoring the symmetries in unimodular gravity

- **Unimodularity in action:** The unimodularity condition can be implemented as an on–shell condition through a lagrange multiplier $\lambda(x)$

$$S = \int d^4x \left[\sqrt{-g} \frac{R}{16\pi G} - \lambda \left(\sqrt{-g} - \epsilon_0 \right) \right]$$

- **Stückelberg-ing the action:** Introduce four Stückelberg fields $\phi^\alpha(x)$, as if we were performing a general coordinate transformation, and let $x^\alpha \to \phi^\alpha(x)$ [2]

$$\int d^4x \lambda \left(\sqrt{-g} - \epsilon_0 \right) \to \int d^4x \lambda \left(\sqrt{-g} - \epsilon_0 \left| J^\alpha{}_\beta \right| \right) \equiv \int d^4x \sqrt{-g} \lambda \left(1 - \epsilon_0 \psi \right)$$

The Stückelberg Jacobian: $\left| J^\alpha{}_\beta \right| \equiv \left| \frac{\partial \phi^\alpha(x)}{\partial x^\beta} \right|$ with $\psi \equiv \frac{\left| J^\alpha{}_\beta \right|}{\sqrt{-g}}$, $\alpha, \beta = 0, \ldots, 3$

- **A new, generalised and Diff-invariant unimodular formulation of GR**

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \lambda f(\psi) - q(\psi) \right]$$

$\rightsquigarrow$ Its easy to see that the equations of motion for the fields λ and ψ **ensure the classical dynamics are the same as** those of GR

[2] See also K. V. Kuchar PRD43, 3332?3344 (1991)

Quantum unimodular gravity: A simple example of equivalence

- Consider the lowest order approximation of the Stückelberged unimodular action [3]

$$S = \int d^4x \left[\sqrt{-g}\frac{R}{16\pi G} - \underbrace{\lambda\left(\sqrt{-g} - \left|J^{\alpha}{}_{\beta}\right|\right)}_{=\lambda(\sqrt{-g}-\partial_\alpha \tau^\alpha)} \right],$$

- Defining the path integral for the theory with only the metric coupled to the sources [4]

$$Z[J] = \int Dg_{\mu\nu} D\lambda D\tau^\mu e^{iS[g,\tau,\lambda]+iS_{\text{ext}}[g,J]}$$

$$S[g,\tau,\lambda] = \int d^4x \left[\sqrt{-g}\frac{R}{16\pi G} - \lambda\left(\sqrt{-g} - \partial_\mu\tau^\mu\right)\right] + \int_{\text{boundary}} d^3x\sqrt{-\gamma}\left[\frac{1}{8\pi G}K - n_\mu\lambda\tau^\mu\right]$$

- Integrating out the field τ^μ and then the lagrange multiplier λ we arrive at the path integral of GR with a cosmological constant

$$Z[J] = \int Dg_{\mu\nu} e^{i\bar{S}_{GR}[g;\lambda_0)+iS_{\text{ext}}[g,J]}$$

$$\bar{S}_{GR}[g;\lambda_0) = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \lambda_0\right] + \int_{\text{boundary}} d^3x\sqrt{-\gamma}\frac{1}{8\pi G}K$$

Quantum Unimodular GR = Quantum GR, provided we make the appropriate assumptions

[3] Reminder: $\left|J^{\alpha}{}_{\beta}\right| = \partial_\alpha\left[4!\delta^{[\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\kappa}\delta^{\delta]}_{\lambda}\phi^\mu J^\nu{}_\beta J^\kappa{}_\gamma J^\lambda{}_\delta\right] \equiv \partial_\alpha\tau^\alpha$

[4] This is essentially the action introduced in M. Henneaux and C. Teitelboim, Phys.Lett. B 222, 195–199 (1989)

Quantum unimodular gravity within the ERG

- We are now interested in understanding the UV structure of the Diff invariant unimodular theory in the Wilsonian approach of the **Exact Renormalisation Group Equation** [5]

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\left(\boldsymbol{\Gamma}_k^{(2)} + \mathbf{R}_k\right)^{-1} k\partial_k \mathbf{R}_k\right],$$

- The starting point is the generating functional, where **now all fields of the theory are coupled to external sources**

$$Z[J] = \int Dg_{\mu\nu} D\phi^\alpha D\lambda \; e^{iS[\Phi_A] + i\int J_A \Phi_A + \Delta S_k}, \qquad \Phi_A = \{g_{\mu\nu}, \lambda, \phi^\alpha\}$$

$$S[\Phi_A] = \int d^4x\sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G} + \lambda(x) f(\psi) + q(\psi)\right], \qquad \psi \equiv \frac{1}{\sqrt{g}}\left|\frac{\partial\phi^\alpha(x)}{\partial x^\beta}\right|$$

- The metric, lagrange multiplier and Stückelberg **fields fluctuate** as

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \sqrt{G_0}\hat{h}_{\mu\nu} + \frac{\sqrt{G_0}}{4}\bar{g}_{\mu\nu}h,$$
$$\lambda = \bar{\lambda} + \sqrt{G_0}\delta\lambda,$$
$$\phi^\alpha = \bar{\phi}^\alpha + G_0^{3/2}\delta\hat{\phi}^\alpha + G_0\bar{\nabla}^\alpha\delta\phi,$$

- The momentum-dependent **Hessian entries** are

$$\Gamma^{(2)}_{hh}, \quad \Gamma^{(2)}_{\hat{h}\hat{h}}, \quad \Gamma^{(2)}_{\phi\phi}, \quad \Gamma^{(2)}_{\phi\lambda}, \quad \Gamma^{(2)}_{h\phi}$$

[5] C. Wetterich Phys. Lett. **B 301**, 90 (1993) | T. R. Morris, Int. J. Mod. Phys. A 9 (1994) 2411.

Quantum unimodular gravity within the ERG

- The (Eucledian) Diff-invariant effective action for the unimodular theory [6]

$$\Gamma_k[g_{\mu\nu}, \lambda, \psi] = -\int d^4x\sqrt{g}\left[Z_G(R-2\Lambda) - \lambda\ f(\psi)\ -\ q(\psi)\ \right] + S_{\text{ghosts}} + S_{\text{gauge fixing}}$$

- The flow equation shall be solved through a polynomial ansatz in the Stückelberg sector

$$f(\psi) = \sum_{i=0}^{N_f} \frac{1}{i!}\rho_i \psi^i \qquad q(\psi) = \sum_{i=1}^{N_q} \frac{1}{i!}\sigma_i \psi^i$$

- Choosing an S_4 background and the optimised regulator function [7]

$$R_k = (k^2 - (-\Box))\Theta(k^2 - (-\Box))$$

we can calculate the beta functions for the dimensionless couplings of the theory

Einstein–Hilbert sector: $\mathrm{k}\partial_k\tilde{\Lambda} = (-2+\eta_{\tilde{\Lambda}})\tilde{\Lambda}, \quad k\partial_k\tilde{G} = (2+\eta_{\tilde{G}})\tilde{G}$

Stückelberg sector: $\mathrm{k}\partial_k\tilde{\rho}_i = (-2+\eta_{\tilde{\rho}_i})\tilde{\rho}_i, \quad k\partial_k\tilde{\sigma}_i = (-4+\eta_{\tilde{\sigma}_i})\tilde{\sigma}_i$

$[\tilde{c} \equiv c/k^d,\ \eta_c \equiv \frac{k\partial_k c}{c}]$

[6] For an interesting and conceptually different study of unimodular gravity within the ERG see A. Eichhorn Class.Quant.Grav., vol. 30, p. 115016 (2013)

[7] D. F. Litim, Phys.Lett., B486, 92–99 (2000)

Quantum unimodular gravity vs quantum GR

Our criteria of comparison between the two theories: The corresponding fixed points and their associated eigenvalues

$$\Gamma_k[g_{\mu\nu}, \lambda, \psi] = -\int d^4x\sqrt{g}\left[Z_G(R-2\Lambda) - \lambda\ f(\psi)\ -\ q(\psi)\ \right] + S_{\text{ghosts}} + S_{\text{gauge fixing}}$$

- The GR truncation: $f(\psi) = 0, \quad q(\psi) = 0$

$$\text{Fixed points:}\quad (\tilde{\Lambda}, \tilde{G}) = (0.193, 0.707)$$
$$\text{Eigenvalues:}\quad (\gamma_\Lambda, \gamma_G) \simeq (-1.99 \pm 3.829i)$$

- The minimal unimodular case: $f(\psi) = \rho_0 + \rho_1\psi, \quad q(\psi) = 0$

$$\text{Fixed points:}\quad (\tilde{\Lambda}, \tilde{G}, \tilde{\rho}_0, \tilde{\rho}_1) = (0.252, 0.520, 0, 0)$$
$$\text{Eigenvalues:}\quad (\gamma_{\tilde{\Lambda},\tilde{G}}, \gamma_{\tilde{\rho}_0}, \gamma_{\tilde{\rho}_1}) = (-2.093 \pm 1.396i, -6.468, -2)$$

- Higher order unimodular sector: $f(\psi) = \sum_{i=0}^{4} \frac{1}{i!}\rho_i\psi^i\ , \qquad q(\psi) = \sum_{i=1}^{4} \frac{1}{i!}\sigma_i\psi^i$

  $\rightsquigarrow$ Fixed point and attractive eigen-values of $(\tilde{\Lambda}, \tilde{G})$ persist and show good quantitative stability

  $\rightsquigarrow$ Stückelberg couplings $\tilde{\rho}_i, \tilde{\sigma}_j$ remain trivial in the UV, while **the associated eigenvalues remain negative as we increase the truncation order**

  $\rightsquigarrow$ The effective actions for GR and the unimodular theory look similar in the UV

$$\Gamma^{\text{Unim.}}\Big|_{k/k_0 \gg 1} \simeq \Gamma^{\text{GR}}\Big|_{k/k_0 \gg 1}$$

The full results

$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\tilde{\rho}_{0*}$	$\tilde{\rho}_{1*}$	$\tilde{\rho}_{2*}$	$\tilde{\rho}_{3*}$	$\tilde{\rho}_{4*}$	$\tilde{\sigma}_{1*}$	$\tilde{\sigma}_{2*}$	$\tilde{\sigma}_{3*}$
0.193	0.707	—	—	—	—	—	—	—	—
0.252	0.520	0	0	—	—	—	—	—	—
0.252	0.520	0	0	0	—	—	—	—	—
0.252	0.520	0	0	0	0	—	—	—	—
0.252	0.520	0	0	0	0	0	—	—	—
0.249	0.528	—	—	—	—	—	—	0	—
0.250	0.525	—	—	—	—	—	—	0	0
0.249	0.527	—	—	—	—	—	—	0	0
0.252	0.520	0	0	—	—	—	0	—	—
0.252	0.520	0	0	—	—	—	0	0	—
$\gamma_{\tilde{\Lambda}}$	$\gamma_{\tilde{G}}$	$\gamma_{\tilde{\rho}_0}$	$\gamma_{\tilde{\rho}_1}$	$\gamma_{\tilde{\rho}_2}$	$\gamma_{\tilde{\rho}_3}$	$\gamma_{\tilde{\rho}_4}$	$\gamma_{\tilde{\sigma}_1}$	$\gamma_{\tilde{\sigma}_2}$	$\gamma_{\tilde{\sigma}_3}$
$-1.475 + 3.043i$	$-1.475 - 3.043i$	—	—	—	—	—	—	—	—
$-2.093 + 1.396$	$-2.093 - 1.396$	-6.468	-2	-2	—	—	—	—	—
$-2.291 + 4.052$	$-2.291 - 4.052$	-6.468	-2	-2	-0.660	—	—	—	—
$-2.291 + 4.052$	$-2.291 - 4.052$	-6.468	-2	-2	-0.660	-1.404	—	—	—
$-2.291 + 4.052$	$-2.291 - 4.052$	-6.468	-2	-2	-0.660	-1.404	-1.89	—	—
$-2.320 + 3.946i$	$-2.320 - 3.946i$	—	—	—	—	—	—	-2.670	—
$-2.301 + 3.964i$	$-2.320 - 3.964i$	—	—	—	—	—	—	-3.040	-3.040
$-2.270 + 3.993i$	$-2.270 - 3.993i$	—	—	—	—	—	—	-3.409	-3.409
$-2.291 + 4.052i$	$-2.291 - 4.052i$	-6.468	-2	—	—	—	-2.660	—	—
$-2.291 + 4.052i$	$-2.291 - 4.052i$	-6.468	-2	—	—	—	-2.660	-3.404	—

Key results and summary

- Unimodular gravity is equivalent to GR at the classical level, and as such does not shed any new light to the cosmological constant problem
- The unimodularity condition can be implemented in a general, Diff-invariant fashion at the level of the action
- Quantum mechanically, equivalence with GR can in principle be established provided any new fields introduced are not coupled to external sources in the generating functional
- Within the Wilsonian approach of the ERG, the two effective actions appear similar in the UV, with only difference the number of relevant Stückelberg couplings increasing with the order of truncation