Aspects of RG Flows in Even Dimensions

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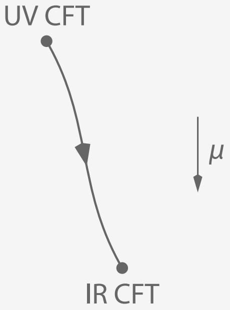
Three deep questions



In relativistic QFTs:

1. Is there a quantity that can tell us what is UV and what is IR (a-theorem)?
2. Are theories with scale invariance necessarily conformal?
3. Are there limit cycles in the RG running?

Essential assumption: Unitarity

Motivation

Phases of QFTs

1. IR-free
   - With mass gap: Exponentially decaying correlators (e.g. confinement)
   - Without mass gap: Trivial correlators (e.g. Abelian Coulomb phase)
2. IR-interacting
   - CFTs: Power-law correlators (e.g. non-Abelian Coulomb phase)
   - Scale-invariant field theories: ?

IR-limits of RG flows

1. Strong coupling (e.g. QCD)
2. Fixed points (CFTs)
3. Limit cycles (?)
4. Ergodic trajectories (?)

a-theorem

Strongest version: Is there a positive definite-tensor G_{ij} in the space of couplings such that $\partial_i V = G_{ij}\beta^j$ for some V?

Strong version: Is there a quantity that decreases monotonically in the flow from the UV to the IR?

Weak version: In the flow between a UV and an IR fixed point, is there a quantity a such that $a_{\mathrm{UV}} > a_{\mathrm{IR}}$?

A monotonically-decreasing quantity was found in $d = 2$ by Zamolodchikov in 1986.

At the RG flow endpoints it becomes the central charge of the corresponding CFT.

The RG flow in $d = 2$ is gradient in conformal perturbation theory.

a-theorem in $d = 4$

4d CFT in curved space: $T^{\mu}{}_{\mu} = aE_4 + cF$

It was suggested by Cardy in 1988 that the coefficient of the Euler term in the trace anomaly, called a, may be the quantity that satisfies a (weak) a-theorem in $d = 4$.

There have been lots of successful checks of Cardy's suggestion over the years.

A nice chain of arguments by Komargodski and Schwimmer proved the weak version of the a-theorem in 2011.

The relevant quantity is indeed a.

In perturbation theory, the strong version of the a-theorem was established by Jack and Osborn in 1990.

The quantity they considered also becomes a at fixed points.

Does scale imply conformal invariance?

$$T^{\mu}_{\ \mu} = \partial_{\mu} V^{\mu} \quad \text{or} \quad 0$$

Scale-invariant *Conformal*

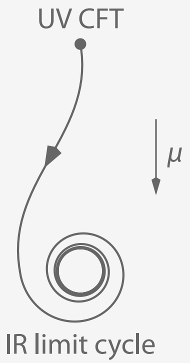
In $d = 2$ Polchinski, following up on the work of Zamolodchikov, showed that scale implies conformal invariance.

In higher spacetime dimensions the situation is still not clear non-perturbatively.

In $d = 4$ within perturbation theory a proof can be found using the results of Jack and Osborn(Fortin, Grinstein & AS), or those of Komargodski and Schwimmer(Luty, Polchinski & Rattazzi).

There has been lots of activity on this subject recently.

Limit cycles



Limit cycles have been suggested as possible endpoints of RG flows in the early '70s by Wilson, but they have never been found in relativistic unitary QFTs.

Limit cycles and ergodic trajectories are actually associated with theories that are scale-invariant but not conformal.(Fortin, Grinstein & AS)

Contents

Work with Jeff Fortin, Ben Grinstein, David Stone, and Ming Zhong
Work in progress with Hugh Osborn

Renormalization

For our considerations we need to extend the usual RG by considering local rescalings of length.

We define the generating functional W by

$$W[g^i] = \ln \int D\varphi \, e^{-S[\varphi, g^i]}.$$

Usual RG:

- A length scale μ^{-1} is introduced to define the theory.
- Rescaling it can be compensated by changing the couplings, as described by the Callan–Symanzik equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta^i \frac{\partial}{\partial g^i} \right) W = 0.$$

Renormalization

Assume now that W is also a function of a background metric,

$$W = W[\gamma_{\mu\nu}, g^i].$$

In the absence of dimensionful couplings, a scale transformation of the metric can be compensated by a corresponding change in μ:

$$\left(\mu \frac{\partial}{\partial \mu} + 2\gamma^{\mu\nu} \frac{\partial}{\partial \gamma^{\mu\nu}} \right) W = 0.$$

Then, the Callan–Symanzik equation implies that

$$\left(2\gamma^{\mu\nu} \frac{\partial}{\partial \gamma^{\mu\nu}} - \beta^i \frac{\partial}{\partial g^i} \right) W = 0.$$

Our aim is to find a local version of this equation.

Local RG

To develop the local RG we imagine that our theory is defined on a manifold, and so the scale μ^{-1} is measured using the metric $\gamma_{\mu\nu}(x)$ of the manifold.

Then, the local RG is defined by the expectation that Weyl rescalings of the metric,

$$\gamma_{\mu\nu}(x) \to e^{-2\sigma(x)}\gamma_{\mu\nu}(x), \qquad \sigma(x): \text{ arbitrary},$$

which induce a change in $\mu^{-1}(x)$, can be compensated by adjusting the couplings, that are now local as well:

$$g^i \to g^i(x).$$

Local RG equation

Generator of Weyl transformations: $\Delta_\sigma^{\mathrm{W}} = 2\int d^d x \sqrt{\gamma}\,\sigma \gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}$.

Variation of the couplings: $\Delta_\sigma^{\beta} = \int d^d x \sqrt{\gamma}\,\sigma \beta^i \frac{\delta}{\delta g^i}$, $\quad \beta^i = \mu\frac{dg^i}{d\mu}$.

The local version of the Callan–Symanzik equation is then

$$\Delta_\sigma^{\mathrm{W}} W = \Delta_\sigma^{\beta} W + \text{terms with derivatives on } \gamma_{\mu\nu}, g^i, \sigma.$$

Since finite operators can be defined via

$$T_{\mu\nu}(x) = 2\frac{\delta S}{\delta\gamma^{\mu\nu}(x)}, \qquad \mathcal{O}_i(x) = \frac{\delta S}{\delta g^i(x)},$$

the local RG equation is equivalent to

$$\gamma^{\mu\nu} T_{\mu\nu} = \beta^i \mathcal{O}_i + \text{terms with derivatives on } \gamma_{\mu\nu}, g^i.$$

This is the general form of the trace anomaly.

Weyl consistency conditions

The algebra of a symmetry broken by quantum corrections constrains the form of the breaking, giving rise to the Wess–Zumino consistency conditions.

The Weyl group is Abelian, so these are very simple here:

$$[\Delta^{\mathrm{W}}_{\sigma} - \Delta^{\beta}_{\sigma}, \Delta^{\mathrm{W}}_{\sigma'} - \Delta^{\beta}_{\sigma'}]W = 0.$$

Nevertheless, they have far-reaching consequences.

These consistency conditions can be decomposed in the basis of the various tensors that appear.

They give rise to both algebraic and differential constraints.

2d c-theorem

In $d = 2$ we start with

$$\Delta_\sigma^{\mathrm{W}} W = \Delta_\sigma^{\beta} W - \int d^2x\sqrt{\gamma}\,\sigma(\tfrac{1}{2}cR - \tfrac{1}{2}\chi_{ij}\partial_\mu g^i \partial^\mu g^j) + \int d^2x\sqrt{\gamma}\,\partial_\mu\sigma\, w_i \partial^\mu g^i,$$

restricted by power-counting and diff-invariance.

There is one consistency condition:

$$\partial_i \tilde{c} = (\chi_{ij} + \partial_i w_j - \partial_j w_i)\beta^j, \qquad \tilde{c} = c + w_i \beta^i,$$

which becomes

$$\beta^i \partial_i \tilde{c} = \chi_{ij}\beta^i\beta^j.$$

The quantities c, χ_{ij}, and w_i have ambiguities, but the consistency condition is invariant under them.

There is choice of the ambiguity such that the “metric” χ_{ij} is positive-definite.

This reproduces Zamolodchikov’s c-theorem.

Strong a-theorem in $d = 4$

In $d = 4$ there are more terms that contribute to the anomaly, and so we get more consistency conditions.

Among them we find again

$$\partial_i \tilde{a} = \tfrac{1}{8}(\chi_{ij} + \partial_i w_j - \partial_j w_i)\beta^j, \qquad \tilde{a} = a + \tfrac{1}{8} w_i \beta^i,$$

from

$$\Delta_\sigma^{\mathrm{W}} W \supset \Delta_\sigma^{\beta} W + \int d^4x \sqrt{\gamma}\, \sigma(a E_4 + \chi_{ij} \partial_\mu g^i \partial_\nu g^j G^{\mu\nu}) + \int d^4x \sqrt{\gamma}\, \partial_\mu \sigma\, w_i\, \partial_\nu g^i G^{\mu\nu}.$$

This metric χ_{ij} can be computed perturbatively, and the leading contribution is found to be positive-definite for the most general classically scale-invariant QFT in $d = 4$. (Jack & Osborn)

Scale vs. conformal invariance in $d = 4$

The dilatation current is

Virial current

$$\mathcal{D}^\mu = x_\nu T^{\mu\nu} - V^\mu.$$

It is conserved if

$$T^\mu_{\ \mu} = \partial_\mu V^\mu.$$

Actually, if $V^\mu = \partial_\nu L^{\mu\nu}$ the theory is still conformal, for then $T_{\mu\nu}$ can be improved to be traceless.

In $d = 4$ theories with scalars, fermions, and gauge fields, the most general virial is

$$V^\mu = Q_{ab}\varphi_a D^\mu \varphi_b - iP_{ij}\bar{\psi}_i \bar{\sigma}^\mu \psi_j,$$

where Q_{ab} is anti-symmetric and P_{ij} anti-Hermitian.

We see that the virial current generates a rotation in field space.

Limit cycles?

Consider a theory with scalars φ_a and the usual quartic coupling. Then,

$$T^\mu{}_\mu = \beta^I \mathcal{O}_I, \qquad V^\mu = Q_{ab}\varphi_a \partial^\mu \varphi_b, \qquad I = (abcd),$$

and the condition for scale-invariance becomes

$$\beta^I = (Qg)^I \Rightarrow -\frac{dg^I}{dt} = (Qg)^I, \qquad t = -\ln \mu.$$

Assuming that Q is constant, this is solved by

$$g^I(t) = g^J(0)(e^{-Qt})_{IJ}.$$

This is an oscillatory solution since Q is anti-symmetric.

Solutions of the above equations with $Q \neq 0$ have been found for 4d QFTs with scalars, fermions, and gauge fields.(Fortin, Grinstein & AS)

It looks like these theories live on limit cycles.

Trace of stress-energy tensor

But is the renormalized stress-energy tensor really given by

$$T^\mu_{\ \mu} = \beta^I \mathcal{O}_I ?$$

This is true only for zero-momentum insertions of $T^\mu_{\ \mu}$. More generally,

$$T^\mu_{\ \mu} = \beta^I \mathcal{O}_I + \partial_\mu J^\mu .$$

If we have scalar fields, for example, $J^\mu = S_{ab}\varphi_a \partial^\mu \varphi_b$, with S_{ab} anti-symmetric.

Using the equations of motion we see then that

$$T^\mu_{\ \mu} = B^I \mathcal{O}_I, \qquad B^I = \beta^I - (Sg)^I .$$

S can be computed in perturbation theory using dim-reg.

The lesson is that a theory is conformal if the B-function is zero.

Scale vs. conformal invariance in $d = 4$

For scale without conformal invariance we have to find solutions to

$$B^I = (Qg)^I \Rightarrow \beta^I - (Sg)^I = (Qg)^I, \qquad (Qg)^I \neq 0.$$

But this is impossible in $d = 4$ perturbation theory!

We have the consistency condition

$$\frac{d\tilde{A}}{dt} = -\frac{1}{8}\chi_{IJ} B^I B^J, \qquad \chi_{IJ}: \text{ perturbatively positive-definite.}$$

A scalar like $\tilde{A}$ cannot change by an orthogonal transformation of the couplings, and so

$$\frac{d\tilde{A}}{dt} = 0 \Rightarrow B^I = 0.$$

This means that whenever $\beta^I = (Rg)^I$, then $(Rg)^I = (Sg)^I$, and thus $(Qg)^I = 0$. (Fortin, Grinstein & AS)

Scale implies conformal invariance in $d = 4$

The results of the previous slides have been verified explicitly at three loops for QFTs in $d = 4$ with scalars, fermions, and gauge fields.(Fortin, Grinstein & AS)

The conclusion is that scale implies conformal invariance perturbatively in $d = 4$.

We do not know if χ_{ij} is positive-definite non-perturbatively, so we can come to our conclusion only within perturbation theory.

In perturbation theory the physically-relevant B-function does not have limit cycles associated with scale invariance.

Strong a-theorem in $d > 4$

Many nontrivial CFTs are known in $d = 6$, although none in $d > 6$.

Very little is known about flows between CFTs in $d > 4$.

A recent study using the methods of Komargodski & Schwimmer did not yield an answer regarding the weak a-theorem.(Elvang et al.)

Weyl consistency conditions can uncover general properties of such flows.

The $d = 6$ case was worked out explicitly recently.(Grinstein, AS & Stone)

Again, a consistency condition analogous to the one in $d = 2, 4$ was discovered.

Consistency condition in any even d

The Euler term is defined in any even $d = 2n$ by

$$E_{2n} = \frac{1}{2^n} R_{i_1 j_1 k_1 l_1} \cdots R_{i_n j_n k_n l_n} \varepsilon^{i_1 j_1 \ldots i_n j_n} \varepsilon^{k_1 l_1 \ldots k_n l_n}.$$

Its Weyl variation in $d = 2n$ is

$$\delta_\sigma(\sqrt{\gamma} E_{2n}) = \sqrt{\gamma}\, H^{\mu\nu} \nabla_\mu \partial_\nu \sigma,$$

where $H_{\mu\nu}$ is the unique two-index tensor of dimension $2(n-1)$ with the properties of the Einstein tensor.(Lovelock)

Crucially, it is covariantly-conserved:

$$\nabla_\nu H^{\mu\nu} = 0.$$

Consistency condition in any even d

The contributions

$$\int d^{2n}x\sqrt{\gamma}\,\sigma\left[(-1)^n aE_{2n} + \sum_p b_p L_p + \tfrac{1}{2}\chi_{ij}\,\partial_\mu g^i \partial_\nu g^j\, H^{\mu\nu}\right] + \int d^{2n}x\sqrt{\gamma}\,\partial_\mu\sigma\, w_i\,\partial_\nu g^i\, H^{\mu\nu},$$

always decouple from everything else and lead to

$$\partial_i\tilde{a} = (\chi_{ij} + \partial_i w_j - \partial_j w_i)\beta^j, \qquad \tilde{a} = a + \mathrm{O}(\beta),$$

and thus to

$$\beta^i\partial_i\tilde{a} = \chi_{ij}\beta^i\beta^j.$$

We do not know, however, if χ_{ij} is positive-definite in general.

The metric in φ^3 theory in $d = 6$

To compute quantities like χ_{ij} one needs to renormalize a theory in curved space with x-dependent couplings.

These quantities are "beta functions" associated with specific counterterms.

A method well-suited for such computations was developed by Jack and Osborn in the early '80s.

It is based on the background-field method and the heat-kernel.

One can compute the effective potential in a manifestly covariant fashion.

Applying this method to multi-flavor φ^3 theory in $d = 6$ we found that χ_{ij} is actually perturbatively negative-definite in this case. At two loops,(Grinstein, AS, Stone & Zhong)

$$\chi_{ij}^{(2)} = -\frac{1}{(64\pi^3)^2}\frac{1}{3240}\delta_{ij}.$$

Things appear different in $d = 6$

The fact that the metric is negative-definite implies that in the flow out of the trivial UV fixed point of φ^3 theory, the quantity $\tilde{a}$ increases.

Regarding the a-theorem, this proves that there is no hope of a strong a-theorem for $\tilde{a}$ in $d = 6$.

It is conceivable that there are other quantities besides $\tilde{a}$ that satisfy a strong a-theorem in $d = 6$.

Such quantities cannot be of the form $\tilde{a} + \mathrm{O}(\beta^2)$.

In φ^3 theory in $d = 6$ there is no IR fixed point in perturbation theory. Thus, we cannot probe the weak version of the a-theorem.

Questions of scale vs. conformal invariance and limit cycles in 6d have not been studied as extensively as in $d = 2, 4$.

Why the negative sign in $d = 6$?

Can we get a better understanding of the sign of χ_{ij} in $d = 6$?

Where does the difference with the $d = 2, 4$ cases come from?

We do not expect the answer to follow from the fact that φ^3 does not have a vacuum.

Our result is perturbative, so let's go very close to the fixed point.

There we can neglect all beta functions to a good approximation.

What form does the anomaly take in that case?

More precisely, we can think of the anomaly on a conformal manifold, where g^i are the couplings of the marginal operators.

The anomaly must be given by terms that appear at fixed points plus conformally-covariant operators acting on g's.

Anomaly in conformal manifold in $d = 2$

In $d = 2$ the Laplacian is a conformally-covariant operator:

$$\nabla^2 \to e^{2\sigma}\nabla^2, \quad \text{when } \gamma^{\mu\nu} \to e^{2\sigma}\gamma^{\mu\nu}.$$

Therefore, the anomaly contribution quadratic in ∂g comes from

$$\Delta_\sigma^{\mathrm{W}} W \supset -\int d^2x\sqrt{\gamma}\,\sigma\,\tfrac{1}{2}G_{ij}\,g^i\nabla^2 g^j.$$

One can show that there is a choice of ambiguity so that G_{ij} is positive-definite.

In any even spacetime dimension there is a unique conformally-covariant "power" of the Laplacian.

It starts as $(-\nabla^2)^{d/2}$ but in $d > 2$ it has more terms.

Anomaly in conformal manifold in $d = 4$

In $d = 4$ the conformally-covariant power of the Laplacian, first written down by Fradkin and Tseytlin but commonly called the Paneitz or Riegert operator, is

$$\Delta_4 = \nabla^2\nabla^2 + \nabla^\mu(4P_{\mu\nu} - 2\gamma_{\mu\nu}\hat{R})\partial^\nu,$$

where $P_{\mu\nu} = \frac{1}{d-2}(R_{\mu\nu} - \gamma_{\mu\nu}\hat{R})$, $\hat{R} = \frac{1}{2(d-1)}R$.

The relevant contribution to the anomaly is

$$\Delta^{\mathrm{W}}_\sigma W \supset \int d^4x\sqrt{\gamma}\,\sigma\,\tfrac{1}{2}G_{ij}\,g^i\Delta_4 g^j.$$

Here we can show that there is a choice of the ambiguity so that G_{ij} is negative-definite.

This is what is required in order to prove the strong a-theorem in conformal perturbation theory in $d = 4$.

Something new in $d = 6$

The conformally-covariant power of the Laplacian here was first found by Branson. It can be written in the form

$$\begin{aligned}\Delta_6 = &-\nabla^2\nabla^2\nabla^2 - 8\nabla^2 P_{\mu\nu}\nabla^\mu\partial^\nu - 8\nabla^\mu\nabla^\nu P_{\mu\nu}\nabla^2 + 6\nabla^2\hat{R}\nabla^2\\ &- \nabla^\mu(8B_{\mu\nu} + 8\nabla_\mu\nabla_\nu\hat{R} + 48P_{\mu\lambda}P_\nu{}^\lambda - 32P_{\mu\nu}\hat{R})\partial^\nu\\ &+ \nabla^\mu(8P_{\rho\lambda}P^{\rho\lambda} - 8\hat{R}^2 + 4\nabla^2\hat{R})\partial_\mu,\end{aligned}$$

where $B_{\mu\nu} = \nabla^\lambda C_{\mu\nu\lambda} - P^{\lambda\rho}W_{\lambda\mu\nu\rho}$, $C_{\mu\nu\lambda} = \nabla_\lambda P_{\mu\nu} - \nabla_\nu P_{\mu\lambda}$.

But in $d = 6$ there are two more conformally-covariant operators:

$$D_1 = \nabla^\mu W_{\mu\lambda\rho\sigma}W_\nu{}^{\lambda\rho\sigma}\partial^\nu \qquad \text{and} \qquad D_2 = \nabla^\mu W_{\kappa\lambda\rho\sigma}W^{\kappa\lambda\rho\sigma}\partial_\mu.$$

For the anomaly this means that

$$\Delta^{\mathrm{W}}_\sigma W \supset \int d^6x\sqrt{\gamma}\,\sigma\,\tfrac{1}{2}\,g^i(G_{1ij}D_1 + G_{2ij}D_2 + G_{3ij}\Delta_6)g^j.$$

Metric in coupling space in $d = 6$

$$\Delta^{\mathrm{W}}_{\sigma} W \supset \int d^6 x \sqrt{\gamma}\, \sigma\, \tfrac{1}{2} g^i (G_{1\,ij} D_1 + G_{2\,ij} D_2 + G_{3\,ij} \Delta_6) g^j .$$

The metric that appears in the a-theorem-like consistency condition in $d = 6$ is not related to $G_{3\,ij}$, but rather to $G_{1\,ij}$.

Although we can show that $G_{3\,ij}$ is positive-definite, there is no argument for positivity of $G_{1\,ij}$ or $G_{2\,ij}$.

At leading order in φ^3 theory we have

$$G_{1,2,3\,ij} = c_{1,2,3} \delta_{ij} .$$

The explicit two-loop heat-kernel result is of this form for some coefficients $c_{1,2,3}$.

This is a good check of the calculation from which we extracted χ_{ij}.

Conclusion and future directions

The a-theorem, the relation between scale and conformal invariance, and the presence of limit cycles in the RG running can be studied in any even spacetime dimension using the local RG.

Starting from 6d, new ingredients appear whose implications have not been explored thoroughly.

Future work:

- Non-perturbative arguments in 4d and 6d.
- Explicit computation of the "metric" χ_{ij} in 6d two-form gauge theory. (Work with Hugh Osborn)
- Holographic understanding of the local RG and the 6d results. (Work with Hong Liu and Elton Yechao Zhu)
- Study of the weak a-theorem in 6d.

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Thank you!