Spontaneous symmetry breaking in fermion systems with functional RG

Andreas Eberlein and Walter Metzner

MPI for Solid State Research, Stuttgart

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Outline

- Introduction
- Coupled fermion-boson flow for a fermionic superfluid
- Fermionic functional RG for a fermionic superfluid
- Fusion of functional RG and mean-field theory

Spontaneous symmetry breaking in condensed matter

Quantum liquids:

- Superfluid $^4\mathrm{He}$ (bosons)
- Superfluid $^3\mathrm{He}$ (fermions)

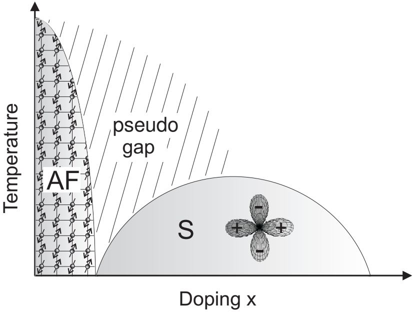
Solids:

- Crystallization

Electrons in solids:

- Magnetism: ferro-, ferri-, antiferro-, ...
- Charge order: density waves, stripes, ...
- Orbital order
- Superconductivity: singlet (s-wave, d-wave), triplet (p-wave)

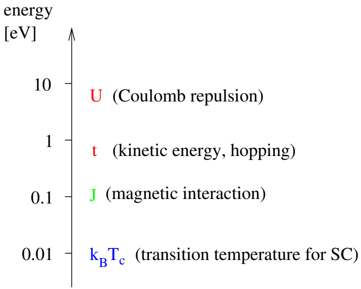
Famous example: CuO_2 high temperature superconductors



- antiferromagnetism in undoped compounds
- d-wave superconductivity at sufficient doping
- Pseudo gap, non-Fermi liquid in "normal" phase at finite T

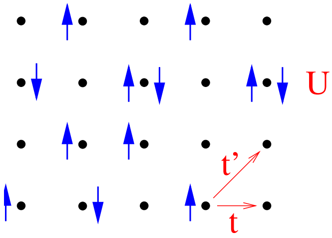
Vast hierarchy of energy scales:

Magnetic interaction and superconductivity generated from kinetic energy and Coulomb interaction



Prototype: Hubbard model

Effective single-band model for CuO_2-planes in HTSC:
(Anderson '87, Zhang & Rice '88)



Hamiltonian $H = H_{kin} + H_I$

$$H_{kin} = \sum_{\mathbf{i},\mathbf{j}} \sum_{\sigma} t_{\mathbf{ij}} \, c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \, n_{\mathbf{k}\sigma}$$

$$H_I = U \sum_{\mathbf{j}} n_{\mathbf{j}\uparrow} n_{\mathbf{j}\downarrow}$$

Antiferromagnetism at/near half-filling for sufficiently large U

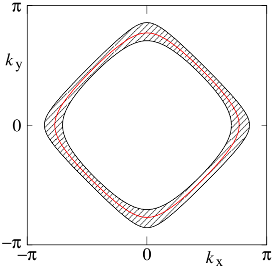
d-wave superconductivity away from half-filling
(perturbation theory, RG, cluster DMFT, variational MC, some QMC)

Fermionic flow equations

Fermi surface singularity at $\omega = 0$, $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu = 0$

Flow parameter: Infrared cutoff $\Lambda > 0$

- Momentum cutoff: $G_0^\Lambda(\mathbf{k}, i\omega) = \dfrac{\Theta(|\xi_{\mathbf{k}}| - \Lambda)}{i\omega - \xi_{\mathbf{k}}}$
- Frequency cutoff: $G_0^\Lambda(\mathbf{k}, i\omega) = \dfrac{\Theta(|\omega| - \Lambda)}{i\omega - \xi_{\mathbf{k}}}$



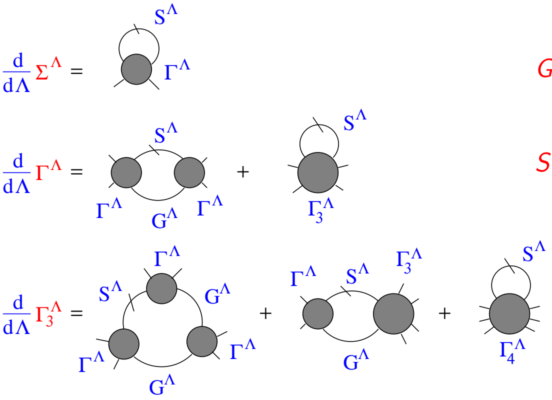
Many other choices: mixed momentum-frequency, smooth cutoff etc.

Initial condition: Λ_0 = band width (momentum) or ∞ (frequency)

Fermionic flow equations

Exact functional flow equation for effective action $\Gamma^\Lambda[\psi, \bar{\psi}]$
(Wetterich 1993; Salmhofer & Honerkamp 2001)

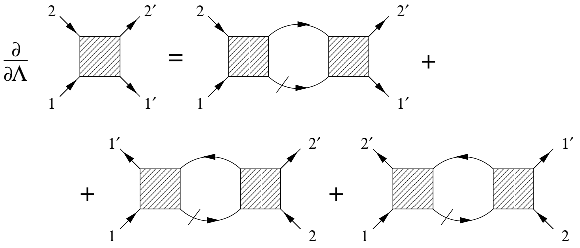
$\Rightarrow$ Exact hierarchy of flow equations for m-particle vertex functions



$$G^\Lambda = \left[(G_0^\Lambda)^{-1} - \Sigma^\Lambda \right]^{-1}$$

$$S^\Lambda = \frac{d}{d\Lambda} G^\Lambda \Big|_{\Sigma^\Lambda \text{ fixed}}$$

Effective 2-particle interaction at 1-loop

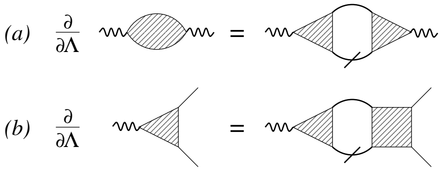


bare G_0^Λ

$\Sigma^\Lambda = 0$

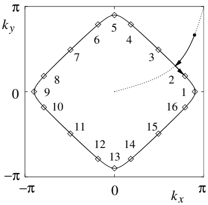
All channels (particle-particle, particle-hole) captured on equal footing.

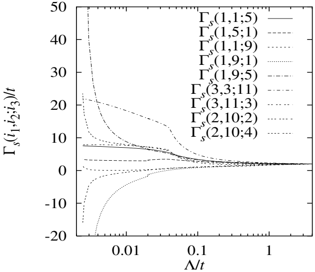
Flow equations for susceptibilities



Effective interaction and susceptibilities in Hubbard model:

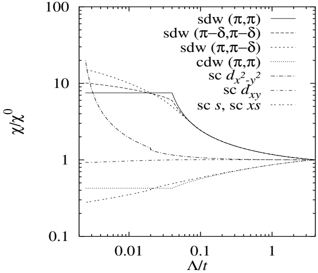
1-loop flow of interactions





Singlet vertex $\Gamma_s^\Lambda(k_1', k_2'; k_1, k_2)$ for various choices of k_1, k_2, k_1'

1-loop flow of susceptibilities



Divergence at critical scale Λ_c indicates instability

$n = 0.984$
$U = t$
$t' = 0$

Zanchi & Schulz '97-'00
Halboth & wm '00
Honerkamp et al. '01

Routes to symmetry breaking

Divergence of effective interaction at scale Λ_c signals instability

$\Rightarrow$ order parameter generated

Routes to spontaneous symmetry breaking in functional RG:

- Hubbard Stratonovich bosonization (Baier, Bick, Wetterich 2004)
- Fermionic flow with order parameter (Salmhofer et al. 2004)
  $\longrightarrow$ *Andreas Eberlein*

Issues:

- Accurate order parameters
- Order parameter fluctuations
- Ward identities – Goldstone mode

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Superfluid state prototype for continuous symmetry breaking with Goldstone mode

This lecture: Focus on ground state ($T = 0$)

Model for fermionic superfluid

Fermionic action with attractive contact interaction $U < 0$

$$\mathcal{S}[\psi, \bar{\psi}] = -\int_{k,\sigma} \bar{\psi}_{k\sigma}(ik_0 - \xi_{\mathbf{k}})\psi_{k\sigma} + U \int_{k,k',q} \bar{\psi}_{q-k\downarrow}\bar{\psi}_{k\uparrow}\psi_{k'\uparrow}\psi_{q-k'\downarrow}$$

$\psi_{k\sigma}$, $\bar{\psi}_{k\sigma}$ Grassmann fields, $k = (k_0, \mathbf{k})$, k_0 Matsubara frequency

Spin-singlet pairing with $U(1)$-symmetry breaking

Complex order parameter $\langle \psi_{k\uparrow}\psi_{-k\downarrow} \rangle$

Hubbard-Stratonovich bosonization

Decouple two-fermion interaction by Hubbard-Stratonovich transformation, introducing a bosonic order parameter field

$$\mathcal{S}[\psi,\bar{\psi},\phi] = -\int_{k,\sigma} \bar{\psi}_{k\sigma}(ik_0 - \xi_{\mathbf{k}})\psi_{k\sigma} - \int_q \phi_q^* \frac{1}{U}\phi_q + \int_{k,q}\left(\bar{\psi}_{q-k\downarrow}\bar{\psi}_{k\uparrow}\phi_q + \psi_{k\uparrow}\psi_{q-k\downarrow}\phi_q^*\right)$$

$U(1)$ symmetry: $\psi \mapsto e^{i\varphi}\psi$, $\bar{\psi} \mapsto e^{-i\varphi}\bar{\psi}$, $\phi \mapsto e^{2i\varphi}\psi$, $\phi^* \mapsto e^{-2i\varphi}\psi$

Exact flow equation for effective action

Add regulator functions R_f^Λ and R_b^Λ for fermions and bosons

$\Rightarrow$ Scale dependent effective action $\Gamma^\Lambda[\psi, \bar{\psi}, \phi]$

Exact flow equation

(Wetterich 1993)

$$\frac{d}{d\Lambda}\Gamma^\Lambda[\psi, \bar{\psi}, \phi] = \frac{1}{2}\mathrm{Str}\frac{\partial_\Lambda \mathbf{R}^\Lambda}{\mathbf{\Gamma}^{(2)\Lambda}[\psi, \bar{\psi}, \phi] + \mathbf{R}^\Lambda}$$

$\mathbf{\Gamma}^{(2)\Lambda}[\psi, \bar{\psi}, \phi]$ matrix of second derivatives w.r.t. fields

Ansatz for effective action

$$\Gamma^\Lambda[\psi,\bar{\psi},\phi] = \Gamma_b^\Lambda[\phi] + \Gamma_f^\Lambda[\psi,\bar{\psi}] + \Gamma_{bf}^\Lambda[\psi,\bar{\psi},\phi]$$

bosons fermions mixed

Two distinct regimes:

- $\Lambda > \Lambda_c$: symmetric regime, no anomalous terms
- $\Lambda < \Lambda_c$: symmetry broken regime, anomalous terms ($\psi\psi$, $\psi\psi\phi$ etc.)

Conditio sine qua non: respect symmetry

Ansatz for effective action: fermions

Only quadratic terms:

$$\Gamma_f^\Lambda[\psi, \bar\psi] = -\int_{k,\sigma} \bar\psi_{k\sigma} \left(i Z_f^\Lambda k_0 - A_f^\Lambda \xi_{\mathbf{k}} \right) \psi_{k\sigma} + \int_{k,\sigma} \left(\Delta^\Lambda \bar\psi_{-k\downarrow} \bar\psi_{k\uparrow} + \Delta^{\Lambda *} \psi_{k\uparrow} \psi_{-k\downarrow} \right)$$

Second term with pairing gap Δ^Λ only in symmetry-broken regime

Z_f^Λ and A_f^Λ *finite* renormalizations – usually discarded ($Z_f^\Lambda = A_f^\Lambda = 1$)

Quartic terms (generated by flow) may be decoupled
during the flow by dynamical bosonization
(Gies & Wetterich 2002, 2004; Floerchinger et al. 2008, 2009)

Ansatz for effective action: fermion-boson interaction

$$\Gamma^{\Lambda}_{bf}[\psi, \bar{\psi}, \phi] = g^{\Lambda} \int_{k,q} \left(\bar{\psi}_{q-k\downarrow} \bar{\psi}_{k\uparrow} \phi_q + \psi_{k\uparrow} \psi_{q-k\downarrow} \phi^*_q \right) + \tilde{g}^{\Lambda} \int_{k,q} \left(\bar{\psi}_{q-k\downarrow} \bar{\psi}_{k\uparrow} \phi^*_{-q} + \psi_{k\uparrow} \psi_{q-k\downarrow} \phi_{-q} \right)$$

Second (anomalous) term only in symmetry-broken regime

$\tilde{g}^{\Lambda}$ small, often neglected ($\tilde{g}^{\Lambda} = 0$)

Renormalization of g^{Λ} small, often neglected ($g^{\Lambda} = 1$)

Other terms (e.g. $\bar{\psi}\psi\phi$, $\bar{\psi}\psi\phi^*\phi$) usually discarded

Ansatz for effective action: bosons

General structure:

$$\Gamma_b^\Lambda[\phi] = \int dx\, U_{\mathrm{loc}}^\Lambda[(\phi(x)] + \text{gradient terms}$$

Simplest (quartic) ansatz for local part:

$$U_{\mathrm{loc}}^\Lambda(\phi) = \tfrac{1}{2}(m_b^\Lambda)^2|\phi|^2 + \tfrac{1}{8}u^\Lambda|\phi|^4 \quad \text{for } \Lambda > \Lambda_c$$

$$U_{\mathrm{loc}}^\Lambda(\phi) = \tfrac{1}{8}u^\Lambda[|\phi|^2 - |\alpha^\Lambda|^2]^2 \quad \text{for } \Lambda < \Lambda_c \text{ (mexican hat)}$$

Ansatz for effective action: boson gradient terms

Simplest (quadratic) ansatz for gradient terms:

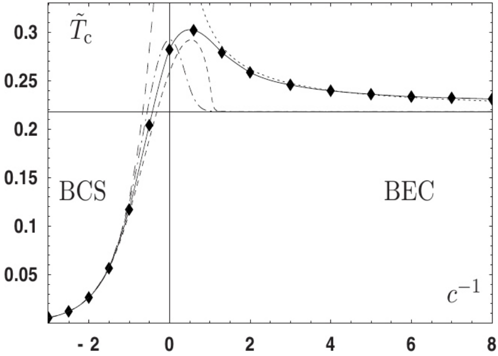
$$\frac{1}{2}\int dx\left[W^\Lambda\phi^*(x)\partial_{x_0}\phi(x) - Z_b^\Lambda|\nabla\phi(x)|^2\right]$$

Birse et al. 2005
Diehl et al. 2007
Krippa 2007

The simplest ansatz, with quartic local and quadratic gradient terms yields decent results for Δ and T_c in three dimensions, not only for weak, but also for strong interactions (from BCS to BEC).

Example: T_c in three dimensions

Critical temperature T_c versus inverse scattering length for fermions in 3D continuum, obtained from relatively simple ansatz for Γ^Λ:



Diehl, Gies, Pawlowski, Wetterich 2007

Decent approximation from weak to strong attraction!

Ansatz for effective action: boson gradient terms

Simplest (quadratic) ansatz for gradient terms:

$$\frac{1}{2}\int dx \left[W^\Lambda \phi^*(x)\partial_{x_0}\phi(x) - Z_b^\Lambda |\nabla\phi(x)|^2 \right]$$

Birse et al. 2005
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The simplest ansatz, with quartic local and quadratic gradient terms yields decent results for Δ and T_c in three dimensions, not only for weak, but also for strong interactions (from BCS to BEC).

But: u^Λ, W^Λ, $(Z_b^\Lambda)^{-1}$ scale to zero for $\Lambda \to 0$ in dimensions $d \leq 3$!

Strong renormalization of longitudinal order parameter fluctuations expected, but transverse fluctuations (Goldstone) should be protected !

$\Rightarrow$ Mission: Save the Goldstone mode!

Ansatz for effective action: boson gradient terms

Goldstone mode protected "by hand" in Strack, Gersch, wm 2008

Goldstone mode protection by symmetry can be achieved by adding quartic gradient term

$$\frac{1}{8}\int dx\, Y^{\Lambda}(\nabla|\phi(x)|^2)^2$$

Strack, PhD thesis 2009
Obert, Husemann, wm 2013

Previously introduced to treat Goldstone mode in $O(N)$ models by Tetradis & Wetterich 1994

Decomposition in longitudinal and transverse fluctuations

Choose bosonic order parameter α^Λ real and positive

Decompose $\phi(x) = \alpha^\Lambda + \sigma(x) + i\pi(x)$, $\quad \phi_q = \alpha^\Lambda \delta_{q0} + \sigma_q + i\pi_q$

with longitudinal (σ) and transverse (π) fluctuations $\quad \Rightarrow$

$$
\begin{aligned}
\Gamma_b^\Lambda[\phi] = & \frac{1}{2}\int_q \left[m_\sigma^2 + Z_\sigma (q_0^2 + \omega_{\mathbf{q}}^2) \right] \sigma_q \sigma_{-q} \\
& + \frac{1}{2}\int_q Z_\pi (q_0^2 + \omega_{\mathbf{q}}^2) \pi_q \pi_{-q} + \int_q W q_0 \pi_q \sigma_{-q} \\
& + \frac{1}{2}\int_{q,p} U(p) \alpha \sigma_p \sigma_q \sigma_{-q-p} + \cdots + \frac{1}{4}\int_{q,q',p} U(p) \sigma_q \sigma_{p-q} \pi_{q'} \pi_{-p-q'}
\end{aligned}
$$

with $m_\sigma^2 = u\alpha^2$, $Z_\pi = Z_b$, $Z_\sigma = Z_b + Y\alpha^2$, and $U(p) = u + Y(p_0^2 + \omega_{\mathbf{p}}^2)$

$\omega_{\mathbf{q}}^2 \sim \mathbf{q}^2$ for small $\mathbf{q}$

Decomposition in longitudinal and transverse fluctuations

$$\Gamma^{\Lambda}_{bf}[\psi, \bar{\psi}, \phi] = g_{\sigma} \int_{k,q} (\bar{\psi}_{q-k\downarrow}\bar{\psi}_{k\uparrow}\sigma_q + \psi_{k'\uparrow}\psi_{q-k'\downarrow}\sigma_{-q}) + i g_{\pi} \int_{k,q} (\bar{\psi}_{q-k\downarrow}\bar{\psi}_{k\uparrow}\pi_q - \psi_{k'\uparrow}\psi_{q-k'\downarrow}\pi_{-q})$$

with $g_{\sigma} = g + \tilde{g}$ and $g_{\pi} = g - \tilde{g}$

Flows for attractive 2D Hubbard model

Nearest neighbor hopping t, dispersion $\epsilon_{\mathbf{k}} = -2t(\cos k_x + \cos k_y)$

Moderate attractive interaction $U = -4t$

Quarter filling (Fermi surface nearly circular)

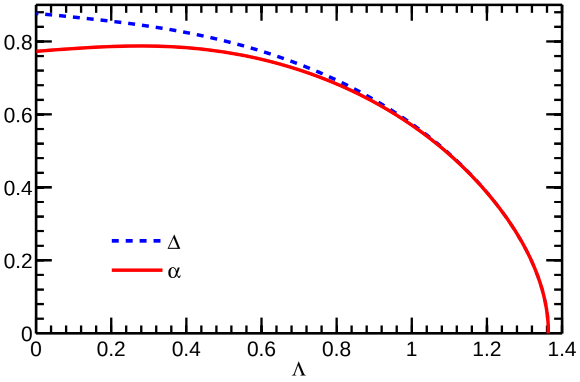
$T = 0$ (ground state)

Regulator functions:

$$R_f(k) = R_f(k_0) = [i\Lambda \mathrm{sgn}(k_0) - ik_0]\,\Theta(\Lambda - |k_0|)$$
$$R_b(q) = R_b(q_0) = Z_b(\Lambda^2 - q_0^2)\,\Theta(\Lambda - |q_0|)$$

Flows for attractive 2D Hubbard model

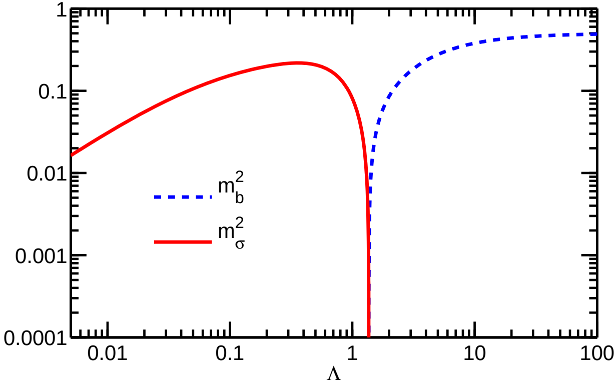
Flow of gap Δ and bosonic order parameter α:



Δ slightly larger than α

Flows for attractive 2D Hubbard model

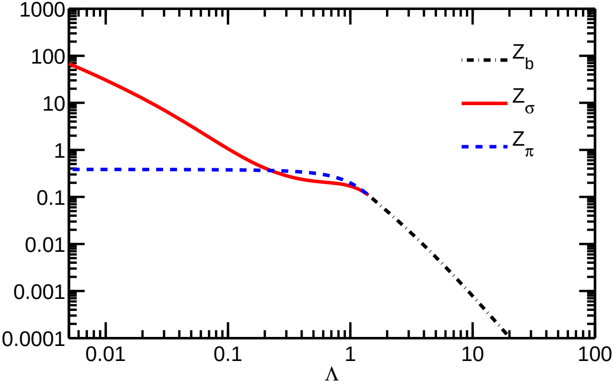
Flow of bosonic masses m_b^2 and m_σ^2:



m_σ^2 vanishes for $\Lambda \to 0$

Flows for attractive 2D Hubbard model

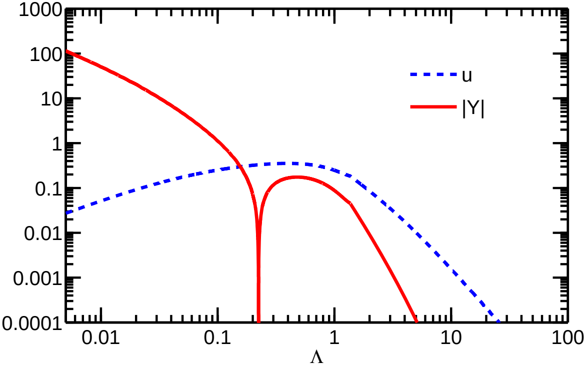
Flow of bosonic Z-factors:



Z_σ diverges, Z_π saturates for $\Lambda \to 0$

Flows for attractive 2D Hubbard model

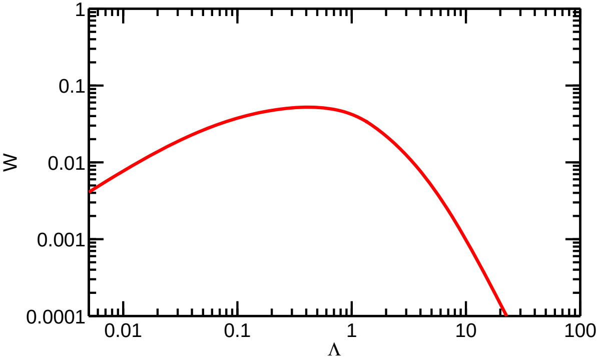
Flow of interaction parameters u and Y:



u vanishes, Y diverges for $\Lambda \to 0$

Flows for attractive 2D Hubbard model

Flow of the σ-π mixing coefficient W:

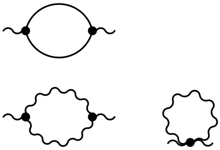


W vanishes for $\Lambda \to 0$

Goldstone mode – cancellations from symmetry

Vanishing Goldstone mass $m_\pi = 0$ conserved by flow?
At first sight, many contributions:

$$\frac{d}{d\Lambda} m_\pi^2 = u\alpha \frac{d\alpha}{d\Lambda} - 2g_\pi^2 \int_k D_\Lambda \left[|G_f(k)|^2 + F_f^2(k) \right] + \tfrac{1}{2} \int_q \left\{ [u + 2U(q)]\, D_\Lambda G_{\pi\pi}(q) + D_\Lambda G_{\sigma\sigma}(q) \right\} - \int_q [U(q)]^2 \alpha^2 D_\Lambda \left[G_{\sigma\sigma}(q) G_{\pi\pi}(q) + G_{\sigma\pi}^2(q) \right]$$



$G_f(k) = -\langle \psi_{k\sigma} \bar\psi_{k\sigma} \rangle$, $F_f(k) = -\langle \psi_{k\uparrow} \psi_{-k\downarrow} \rangle$

$G_{\sigma\sigma}(q) = \langle \sigma_q \sigma_{-q} \rangle$, $G_{\pi\pi}(q) = \langle \pi_q \pi_{-q} \rangle$, $G_{\pi\sigma}(q) = \langle \pi_q \sigma_{-q} \rangle$

Derivative D_Λ acts only on regulator function in propagators

Goldstone mode – cancellations from symmetry

Bosonic fluctuation contributions cancel $\Rightarrow$

$$\frac{d}{d\Lambda} m_\pi^2 = 2\left(\frac{g_\sigma}{\alpha} - \frac{g_\pi^2}{\Delta}\right) \int_k D_\Lambda F_f(k)$$

Ward identity from $U(1)$-symmetry:

$\Delta = g\alpha - \tilde{g}\alpha^* = g_\pi \alpha$ for real α

case $\tilde{g} = 0$:
Bartosch, Kopietz, Ferraz 2009

$$\Rightarrow \frac{d}{d\Lambda} m_\pi^2 = \frac{2}{\alpha}\left(g_\sigma - g_\pi\right) \int_k D_\Lambda F_f(k) \quad \Rightarrow$$

$$\frac{d}{d\Lambda} m_\pi^2 = 0 \text{ iff } g_\sigma = g_\pi$$

Goldstone mass vanishes

Goldstone mode – cancellations from symmetry

Setting $g_\sigma = g_\pi$ yields further cancellations in other quantities such that

- Ward identity $\Delta = g_\pi \alpha$ consistent with flow of Δ, g_π and α
- Z_π finite for $\Lambda \to 0$
- $W/m_\sigma^2 \to C$ finite for $\Lambda \to 0$

$$\Rightarrow \quad G_{\pi\pi}(q) \sim \frac{1}{Z_\pi(q_0^2 + \omega_{\mathbf{q}}^2)} \qquad G_{\pi\sigma}(q) \sim \frac{C q_0}{Z_\pi(q_0^2 + \omega_{\mathbf{q}}^2)}$$

m_σ^2, W, and Z_σ^{-1} vanish for $\Lambda \to 0$ (proportional to Λ in 2D)

$G_{\sigma\sigma}(q)$ exhibits anomalous scaling

Full agreement with IR behavior known for interacting bosons (e.g. Pistolesi et al. 2004)

Goldstone mode – cancellations from symmetry

Not yet happy?

Exact flow would yield $g_\sigma \neq g_\pi$

Consistent truncation with $g_\sigma \neq g_\pi$ requires inclusion of two-fermion-two-boson vertices (Ward identity)

Summary (symmetry-breaking via bosonization)

- fRG with Hubbard-Stratonovich bosonization ideal framework to treat order parameter fluctuation effects
- A simple truncation of the effective action captures all singularities associated with the Goldstone boson in fermionic superfluids (Obert, Husemann, wm 2013)
- Many other problems of symmetry-breaking treated by fRG with HS-fields, mostly by Heidelberg group: antiferromagnetism, d-wave superconductivity, Kosterlitz-Thouless transition, ...