Fermions in gravity with local spin-base invariance

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Motivation

aim at a theory of quantized matter and quantized forces

matter is composed of fermions $\Rightarrow \mathcal{D}\psi\mathcal{D}\bar{\psi}$ ✓

SM forces mediated via gauge field $\Rightarrow \mathcal{D}\mathcal{A}_\mu^a$ ✓

gravity encoded in spacetime curvature $\Rightarrow \mathcal{D}?$ ✗

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$\Rightarrow$ vierbein formalism is mandatory

$$e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}, \quad \nabla_\mu e_\nu{}^a = 0$$

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Issue I - Clifford algebra

Clifford algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$$

multiply with vierbein ${e_\mu}^a$:

$$\{{e^\mu}_a\gamma_\mu, {e^\nu}_b\gamma_\nu\} = 2{e^\mu}_a{e^\nu}_b g_{\mu\nu}\mathrm{I} = 2\eta_{ab}\mathrm{I}$$

$\Rightarrow {\gamma_{(\mathrm{f})}}_a := {e^\mu}_a\gamma_\mu$ are usual flat spacetime gamma matrices?

then they would satisfy $\partial_\mu {\gamma_{(\mathrm{f})}}_a = 0$

BUT: there are γ_μ, where one cannot remove the spacetime dependence with a vierbein

Issue II - Coordinate transformations

Lorentz transformations $\Lambda^a{}_b$ in flat space: $\Lambda^c{}_a \Lambda^d{}_b \eta_{cd} = \eta_{ab}$

nice property of the flat gammas: $\Lambda^b{}_a \gamma_{(\mathrm{f})\,b} = \mathcal{S}_{\mathrm{Lor}} \gamma_{(\mathrm{f})\,a} \mathcal{S}_{\mathrm{Lor}}^{-1}$

BUT: no general rule for coordinate transformations

e.g.: $x^3 \to \frac{1}{\alpha} x^3 \Rightarrow \eta'_{ab} = \mathrm{diag}(-1, 1, 1, \alpha^2)$

on the other hand $(\mathcal{S} \gamma_{(\mathrm{f})\,3} \mathcal{S}^{-1})^2 = \mathrm{I} \neq \alpha^2 \mathrm{I}$

Spin-base invariance

Dirac structure:

- Clifford algebra (irreducible representation):
  $\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathrm{I}, \quad \gamma_\mu \in \mathbb{C}^{d_\gamma \times d_\gamma}, \quad d_\gamma = 2^{\lfloor d/2 \rfloor}$

- Dirac conjugation with spin metric h:
  $\bar{\psi} = \psi^\dagger h, \quad |\det h| = 1$

perform coordinate transformation $\gamma_\mu \to \gamma'_\mu$

$$\{\gamma'_\mu, \gamma'_\nu\} = 2 g'_{\mu\nu} \mathrm{I} = 2 \tfrac{\partial x^\rho}{\partial x'^\mu} \tfrac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda} \mathrm{I} = \{ \tfrac{\partial x^\rho}{\partial x'^\mu} \gamma_\rho, \tfrac{\partial x^\lambda}{\partial x'^\nu} \gamma_\lambda \}$$

most general solution $\gamma'_\mu = \pm \frac{\partial x^\rho}{\partial x'^\mu} \mathcal{S} \gamma_\rho \mathcal{S}^{-1}$, $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$

have two independent coordinate transformations

- diffeomorphisms (change of spacetime base):
  $\gamma_\mu \to \gamma'_\mu = \frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho$
  $\psi \to \psi' = \psi$
  $h \to h' = h$

- spin base transformations (change of spin base):
  $\mathcal{S}_\varphi = \mathrm{e}^{\mathrm{i}\varphi}\mathcal{S} \in \mathrm{SB}(d_\gamma) = \{\mathrm{e}^{\mathrm{i}\varphi}\mathcal{S} : \mathrm{e}^{\mathrm{i}\varphi} \in \mathrm{U}(1), \mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})\}$
  $\gamma_\mu \to \gamma'_\mu = \pm\mathcal{S}_\varphi \gamma_\mu \mathcal{S}_\varphi^{-1}$
  $\psi \to \mathcal{S}_\varphi \psi$
  $h \to \pm(\mathcal{S}_\varphi^\dagger)^{-1} h \mathcal{S}_\varphi^{-1}$

covariant derivative:

- linearity:
  $$\nabla_\mu(\psi_1 + \psi_2) = \nabla_\mu\psi_1 + \nabla_\mu\psi_2$$
- product rule:
  $$\nabla_\mu\psi\bar{\psi} = (\nabla_\mu\psi)\bar{\psi} + \psi(\nabla_\mu\bar{\psi})$$
- spin-base covariance:
  $$\nabla_\mu\bar{\psi} = \overline{\nabla_\mu\psi}$$
- coordinate covariance:
  $$\nabla_\mu(\bar{\psi}\gamma^\nu\psi) = D_\mu(\bar{\psi}\gamma^\nu\psi) \equiv \partial_\mu(\bar{\psi}\gamma^\nu\psi) + \Gamma^\nu_{\mu\kappa}(\bar{\psi}\gamma^\kappa\psi),$$
  $$\Gamma^\nu_{\mu\kappa} = \left\{ {\nu \atop \mu\kappa} \right\} + K^\nu{}_{\mu\kappa}, \quad \left\{ {\nu \atop \mu\kappa} \right\} = \frac{1}{2}g^{\nu\rho}(\partial_\mu g_{\kappa\rho} + \partial_\kappa g_{\mu\rho} - \partial_\rho g_{\mu\kappa})$$

reality of action:

- mass term:
  $(\bar{\psi}\psi)^* = \bar{\psi}\psi$

- kinetic term:
  $\int \mathrm{d}^d x \sqrt{-g}\ (\bar{\psi}\nabla\!\!\!/\psi)^* = \int \mathrm{d}^d x \sqrt{-g}\ \bar{\psi}\nabla\!\!\!/\psi, \quad \nabla\!\!\!/\psi = \gamma^\mu \nabla_\mu \psi$

Spin metric and spin connection

construct spin metric h and spin covariant derivative ∇_μ

spin metric is implicitly determined through the γ^μ

Spin metric

$$\gamma_\mu^\dagger = -h\gamma_\mu h^{-1}, \quad h^\dagger = -h, \quad |\det h| = 1$$

spin covariant derivative:

$$\nabla_\mu\psi = \partial_\mu\psi + \hat\Gamma_\mu\psi + \Delta\Gamma_\mu\psi + \mathrm{i}\mathcal{A}_\mu \,\Big|\, D_\mu v^\alpha = \partial_\mu v^\alpha + \left\{ {\alpha \atop \mu\beta} \right\} v^\beta + K^\alpha{}_{\mu\beta} v^\beta$$

canonical part of the spin connection $\hat\Gamma_\mu$

$$D_{(\mathrm{LC})\mu}\gamma^\nu = \partial_\mu\gamma^\nu + \left\{ {\nu \atop \mu\kappa} \right\}\gamma^\kappa = -[\hat\Gamma_\mu, \gamma^\nu], \quad \mathrm{tr}\,\hat\Gamma_\mu = 0$$

for example in $d = d_\gamma = 4$:

$$\hat\Gamma_\mu = p_\mu\gamma_* + v_\mu{}^\alpha\gamma_\alpha + a_\mu{}^\alpha\gamma_*\gamma_\alpha + t_\mu{}^{\alpha\beta}[\gamma_\alpha, \gamma_\beta],$$

$$\gamma_* = -\frac{\mathrm{i}}{4!}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_4}\gamma^{\mu_1}\dots\gamma^{\mu_4}$$

$$p_\mu = \frac{1}{32}\mathrm{tr}(\gamma_*\gamma_\alpha\partial_\mu\gamma^\alpha), \quad v_\mu{}^\alpha = \frac{1}{48}\mathrm{tr}([\gamma^\alpha, \gamma_\beta]\partial_\mu\gamma^\beta),$$

$$a_\mu{}^\alpha = \frac{1}{8}\mathrm{tr}(\gamma_*\partial_\mu\gamma^\alpha), \quad t_{\mu\alpha}{}^\beta = -\frac{1}{32}\mathrm{tr}(\gamma_\alpha\partial_\mu\gamma^\beta) - \frac{1}{8}\left\{ {\beta \atop \mu\alpha} \right\}$$

spin torsion $\Delta\Gamma_\mu$:

$$0 = [\Delta\Gamma_\mu, \gamma^\mu], \quad \Delta\Gamma_\mu = -h^{-1}\Delta\Gamma_\mu^\dagger h$$

for example in $d = d_\gamma = 4$:
spin torsion carries 45 real parameters, but only 11 remain within the Dirac operator $\nabla\!\!\!/$:

$$\bar{\psi}\gamma^\mu\Delta\Gamma_\mu\psi = \mathscr{M}\bar{\psi}\psi - \mathscr{A}_\mu\bar{\psi}\mathrm{i}\gamma_*\gamma^\mu\psi - \mathscr{F}_{\mu\nu}\bar{\psi}\tfrac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]\psi$$

$\mathscr{M}$: mass/scalar field

$\mathscr{A}_\mu$: axial vector field

$\mathscr{F}_{\mu\nu}$: anti-symmetric tensor field to recover vierbein formalism, set ${\gamma^{(e)}}_\mu = {e_\mu}^a {\gamma^{(\mathrm{f})}}_a$ and find

$$\hat{\Gamma}_\mu + \Delta\Gamma_\mu = \frac{1}{8}{\omega_\mu}^{ab}[{\gamma^{(\mathrm{f})}}_a, {\gamma^{(\mathrm{f})}}_b]$$

$$\Delta\Gamma_\mu = \frac{1}{8}{K^\alpha}_\mu{}^\beta[{\gamma^{(e)}}_\alpha, {\gamma^{(e)}}_\beta]$$

the vierbein formalism gives an additional constraint on spacetime torsion: $\quad {K^\alpha}_{\alpha\mu} = 0$

or it violates: $\quad \int \mathrm{d}^d x \sqrt{-g}\ (\bar{\psi}\nabla\!\!\!/\,\psi)^* = \int \mathrm{d}^d x \sqrt{-g}\ \bar{\psi}\nabla\!\!\!/\,\psi$

Path integral

can construct action S if γ^μ are known

naive way: $\int \mathcal{D}\gamma\, \mathrm{e}^{\mathrm{i}S}$ (and fermions, gauge fields, ...)

but the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$ prohibits arbitrary variations of γ_μ

$\Rightarrow$ determine degrees of freedom from Clifford algebra

Weldon theorem [Weldon '01]

$$\delta\gamma^\mu = \frac{1}{2}(\delta g^{\mu\nu})\gamma_\nu + [\delta\mathcal{S}_\gamma, \gamma^\mu], \ \mathrm{tr}\,\delta\mathcal{S}_\gamma = 0$$

$\delta g_{\mu\nu}$: metric fluctuations

$\delta\mathcal{S}_\gamma$: spin-base fluctuations, $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$

outlook

- existence of gamma matrices on all metrizable manifolds?
- construction of an action from the field strength $\Phi_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$
- metric based path integral quantization of gravity in the presence of fermions

summary

- impose full nontrivial symmetry of Clifford algebra $\Rightarrow$ spinbase transformations: $\mathrm{SB}(d_\gamma)$
- impose *natural* conditions on ∇_μ
- spin metric h and canonical part of the spin connection $\hat{\Gamma}_\mu$ are determined through γ_μ
- vierbein formalism can be recovered

Thank you for your attention!