Weak Renormalization Group Analysis of the Dynamical Chiral Symmetry Breaking

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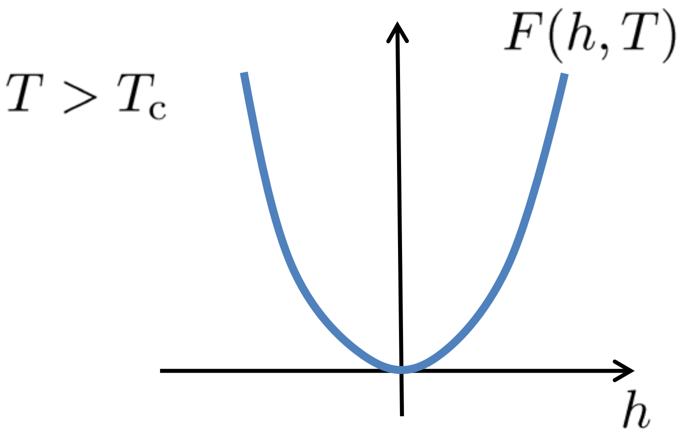
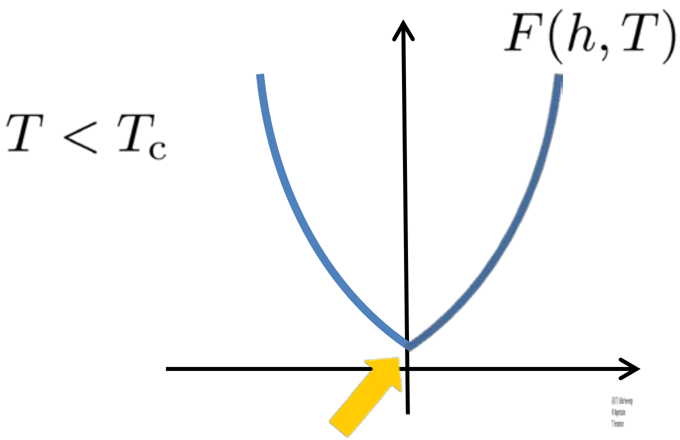
Introduction

Spontaneous symmetry breaking in 2D Ising model

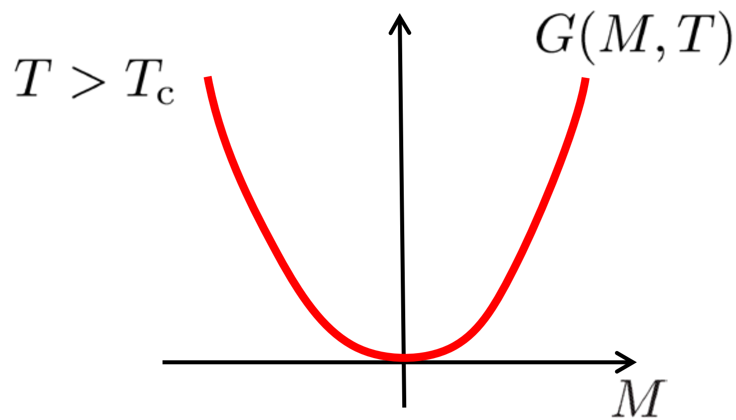
$F(h, T)$: Helmholtz free energy
h : External field
T: Temperature

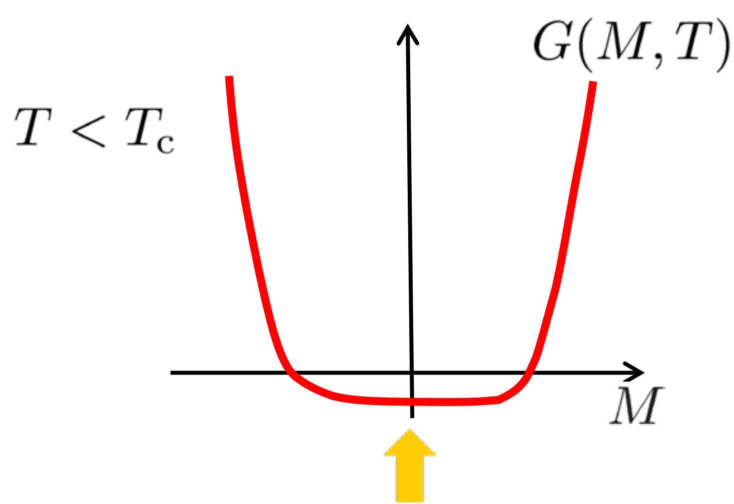




Nondifferentiable point
(2nd order phase transition)

$G(M, T)$: Gibbs free energy
M: Expectation value of magnetization
T: Temperature



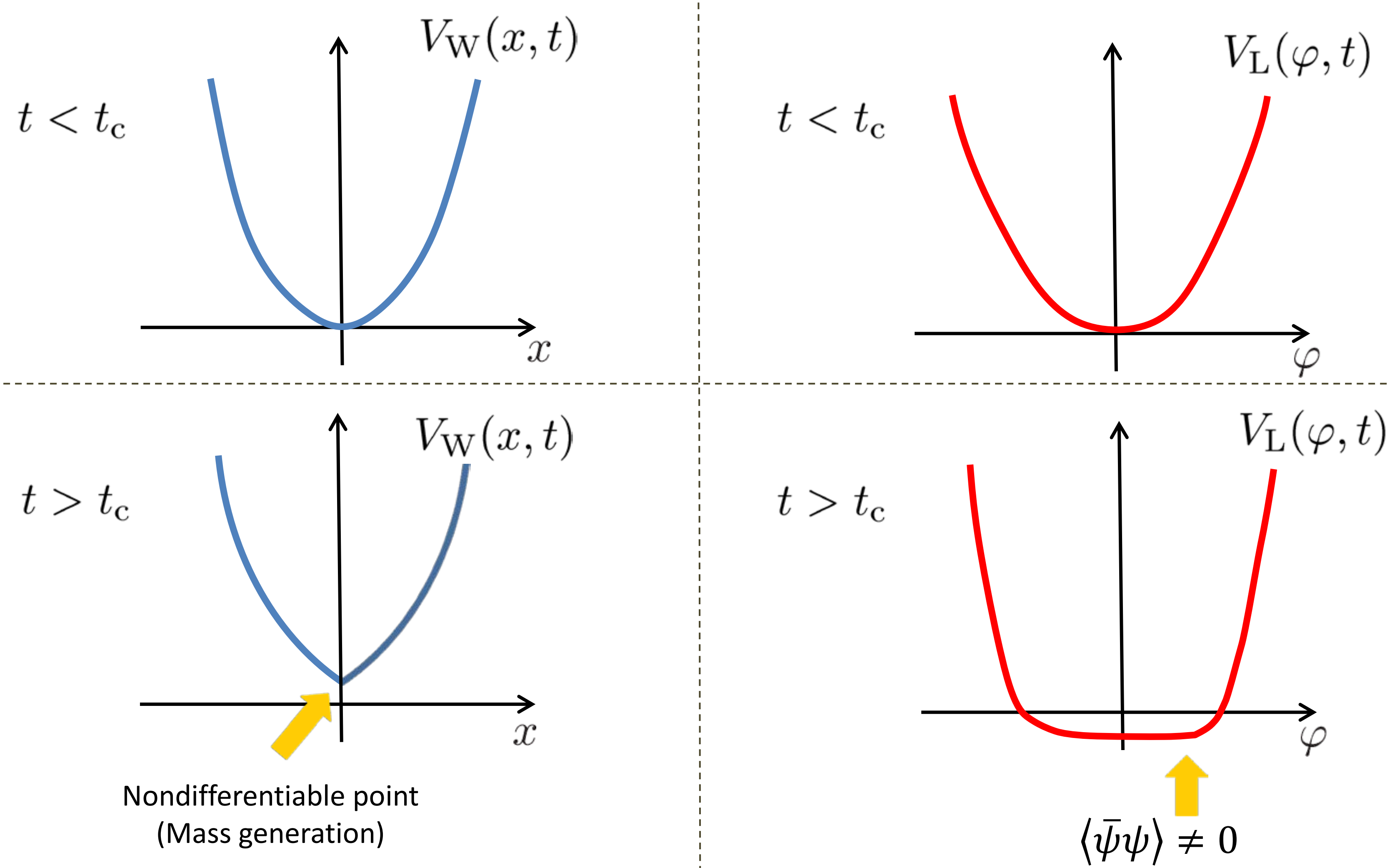


Flat bottom

Dynamical chiral symmetry breaking in the Nambu－Jona-Lasinio model (0 density)

$V_{\mathrm{W}}(x,t)$: Wilsonian effective potential
x : Bilinear fermion operator $\bar{\psi}\psi$
t: Renormalization scale parameter

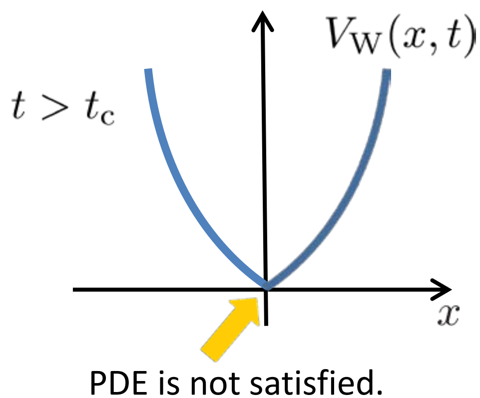
$V_{\mathrm{L}}(\varphi,t)$: Legendre effective potential
φ: Chiral condensate $\langle\bar{\psi}\psi\rangle$
t: Renormalization scale parameter



- Wilsonian effective potential $V_{\mathrm{W}}(x,t)$ is obtained as the solution of the nonperturbative renormalization group equation (NPRGE).

$$\frac{\partial V_{\mathrm{W}}(x,t)}{\partial t} + f\left(\frac{\partial V_{\mathrm{W}}}{\partial x}, t\right) = 0$$

- The singular solution ($t > t_{\mathrm{c}}$) is not the “usual solution ” of the partial differential equation (PDE).
- Usual difference methods and the Taylor expansion don’t work above t_{c} .



In order to authorize such a singular solution, we introduce the "weak solution" as the mathematically extended notion of solution, which has some nondifferentiable points.

K-I. Aoki, S-I. Kumamoto and D. Sato PTEP 2014 043B05

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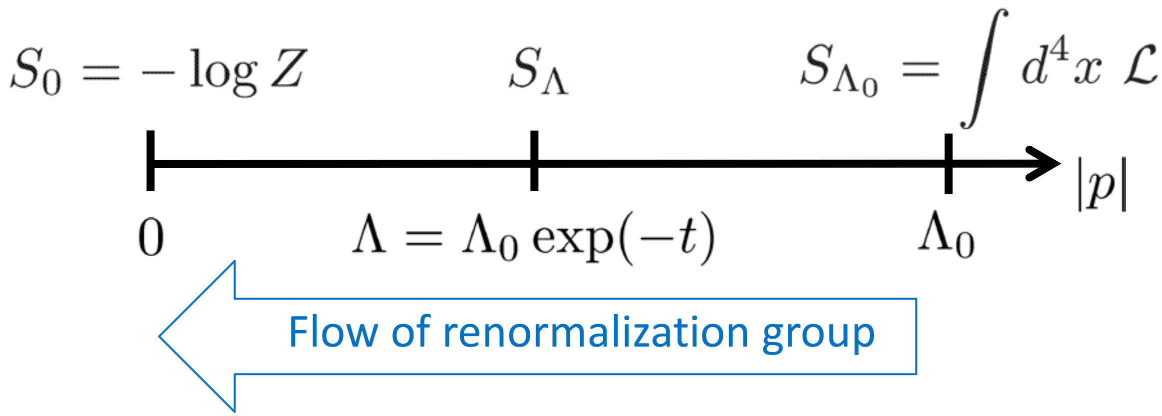
1. Nonperturbative renormalization group equation

Partition function

$$Z = \int_{0 \le |p| \le \Lambda} \mathcal{D}\phi \quad \exp(-S_\Lambda)$$

Wilsonian effective action



Nonperturbative renormalization group equation (Wegner-Houghton equation)

$$\frac{\partial S_\Lambda}{\partial t} = \frac{1}{2\delta t} \int_{(\Lambda - \Lambda\delta t) < |p| < \Lambda} \frac{d^4 p}{(2\pi)^4} \left[\mathrm{Tr} \ln \left(\frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} \right) - \frac{\delta S_\Lambda}{\delta\phi(p)} \left(\frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} \right)^{-1} \frac{\delta S_\Lambda}{\delta\phi(-p)} \right]$$

This functional differential equation (FDE) can not be solved.
We reduce the FDE to a partial differential equation by the LPA.

Local potential approximation for fermions

$$S_\Lambda \simeq \int d^4y\ [\bar{\psi} i \partial\!\!\!/ \psi - V_{\mathrm{W}}(x,t)]$$

Wilsonian effective potential

$$x \equiv \bar{\psi}\psi, \quad t \equiv -\log \frac{\Lambda}{\Lambda_0}$$

Nonperturbative renormalization group equation (1st order PDE)

$$\frac{\partial V_{\mathrm{W}}(x,t)}{\partial t} + f\,(M,t) = 0 \qquad \left(M \equiv \frac{\partial V_{\mathrm{W}}}{\partial x} \right)$$

$$f(M,t) \equiv -\frac{e^{-3t}}{\pi^2}\left[\theta\left(e^{-2t}+M^2-\mu^2\right)\sqrt{e^{-2t}+M^2}+\theta\left(-e^{-2t}-M^2+\mu^2\right)\mu\right]$$

Initial condition
(Finite density NJL model)

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g}{2}(\bar{\psi}\psi)^2 - \mu\psi^\dagger\psi - m_0\bar{\psi}\psi$$

$$V_{\mathrm{W}}(x,0) = 2\pi^2 g x^2 + m_0 x$$

We reduce the PDE to a system of ordinary differential equations (ODEs).

The method of characteristics (PDE ⇒ system of ODEs)

Change of variables

NPRGE

$$\frac{\partial V_{\mathrm{W}}(x,t)}{\partial t} + f(M,t) = 0$$

($f(M,t)$: concave in M)

$$S(x,t) \equiv -V_{\mathrm{W}}(x,t)$$
$$p \equiv -M$$
$$H(p,t) \equiv -f(M,t)$$

Hamilton-Jacobi type equation

$$\frac{\partial S(x,t)}{\partial t} + H(p,t) = 0$$

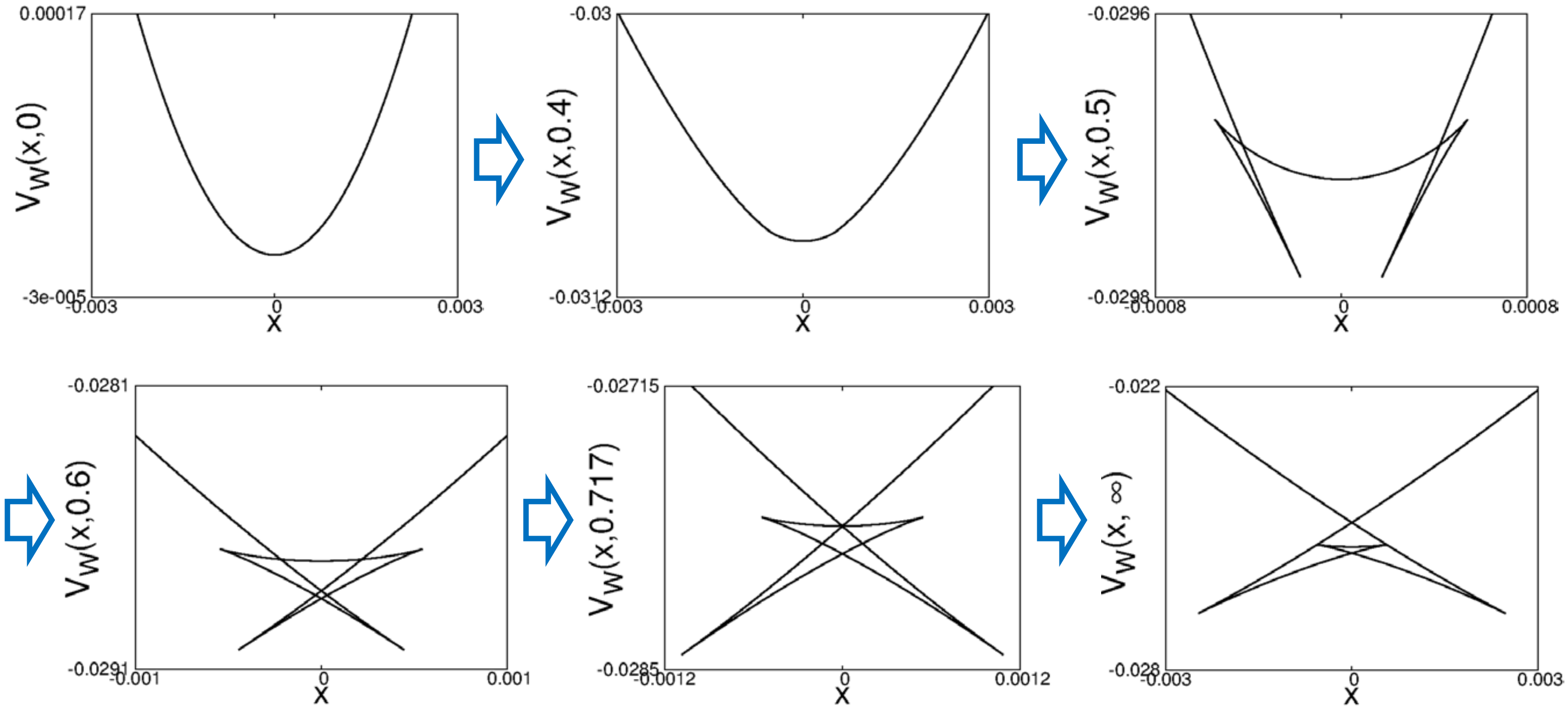
($H(p,t)$: convex in p)

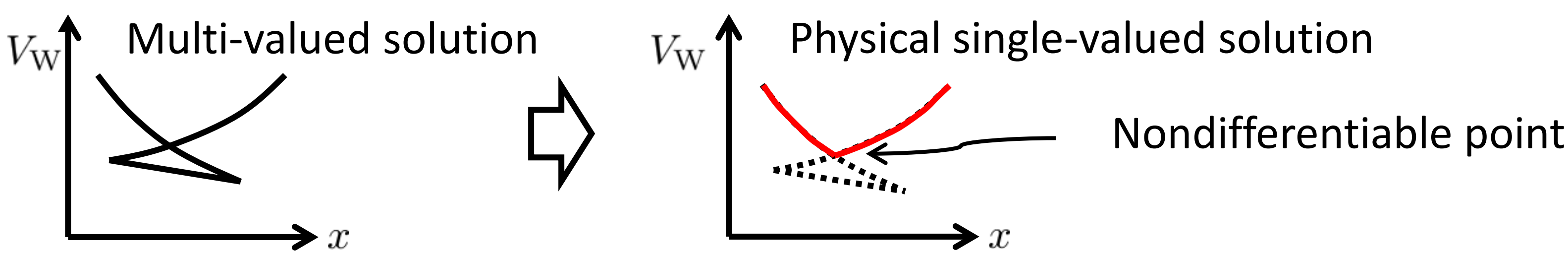
Characteristic equations

$$\begin{cases} \dfrac{dx_{\mathrm{c}}(t)}{dt} = \left.\dfrac{\partial H(p,t)}{\partial p}\right|_{p=p_{\mathrm{c}}(t)} \\[2ex] \dfrac{dp_{\mathrm{c}}(t)}{dt} = -\left.\dfrac{\partial H(p,t)}{\partial x}\right|_{x=x_{\mathrm{c}}(t)} \\[2ex] \dfrac{dS(x_{\mathrm{c}}(t),t)}{dt} = p_{\mathrm{c}}(t)\dot{x}_{\mathrm{c}}(t) - H(p_{\mathrm{c}}(t),t) \end{cases}$$

Canonical equations

t-evolution of Wilsonian effective potential $V_w(x,t)$
($m_0 = 0$, $g = 1.7 \times g_c$, $\mu = 0.7$, $t = 0$, 0.4, 0.5, 0.6, 0.717, 10)





We introduce the weak solution which is a single-valued and singular solution.

2. Weak solution

Hamilton-Jacobi equation ($S(x,t)$: singular solution)

$$\frac{\partial S(x,t)}{\partial t} + H\left(\frac{\partial S}{\partial x}, t\right) = 0$$

Add the viscosity term
(second derivative with respect to x).

Viscous Hamilton-Jacobi equation ($S^{\epsilon}(x,t)$: regularized smooth solution)

$$\frac{\partial S^{\epsilon}(x,t)}{\partial t} + H\left(\frac{\partial S^{\epsilon}}{\partial x}, t\right) = \epsilon \frac{\partial^2 S^{\epsilon}(x,t)}{\partial x^2} \quad (\ \epsilon > 0\ )$$

viscosity term

Weak solution of H-J eq.
(Viscosity solution)

$$S(x,t) \equiv \lim_{\epsilon \to 0} S^{\epsilon}(x,t)$$

M.G. Crandall and P.-L. Lions, *Trams. Amer. Math. Soc.* **277**, 1 (1983).

How to calculate viscosity solution

There are two major methods of calculation for viscosity solution.

- Vanishing viscosity method (Definition of viscosity solution)
  1. Calculate the regularized smooth solution $S^{\epsilon}(x,t)$.
  2. Take the limit

$$S(x,t) \equiv \lim_{\epsilon \to 0} S^{\epsilon}(x,t).$$

- Convert to the optimal control problem
  1. Replace the initial value problem of Hamilton-Jacobi equation with a completely different problem which is called the “optimal control problem”.
  2. Calculate the value function which is equivalent to the viscosity solution.

⇨ We solve the optimal control problem to calculate the viscosity solution.

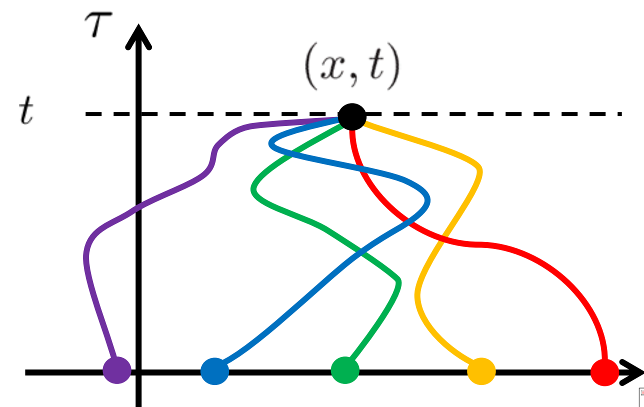
Optimal control problem

Cost functional: $J[\chi] \equiv \int_0^t L(\chi(\tau), \dot{\chi}(\tau), \tau)\, d\tau \Big|_{\chi(t)=x} + S_0(\chi(0))$

Lagrangian   Initial action

Value function

$$S(x,t) \equiv \min_{\chi(\cdot)} J[\chi] \ (= J[\chi^*])$$



(Optimal control problem is to find the path $\chi^*(\tau)$ which minimizes $J[\chi]$.)

It is known that the value function is equivalent to the viscosity solution of the Hamilton-Jacobi equation.

$$\frac{\partial S(x,t)}{\partial t} + H\left(x, \frac{\partial S}{\partial x}, t\right) = 0$$

$$H(x,p,t) \equiv \max_v [vp - L(x,v,t)]$$

$$S(x,0) = S_0(x)$$

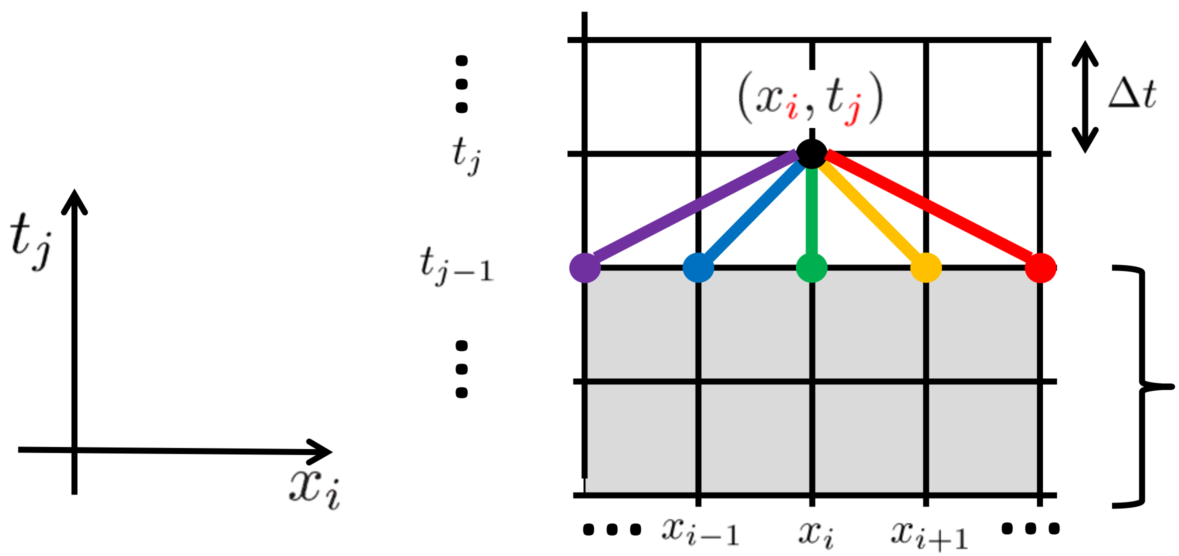
Dynamic programming (Method to calculate the value function)

Optimality condition

$$S(x,t) = \min_{x} \left[\int_{t-\Delta t}^{t} L(x(\tau), \dot{x}(\tau), \tau)\, d\tau + S\big(x(t-\Delta t), t-\Delta t\big) \right]$$

Discretize (x, t) plane

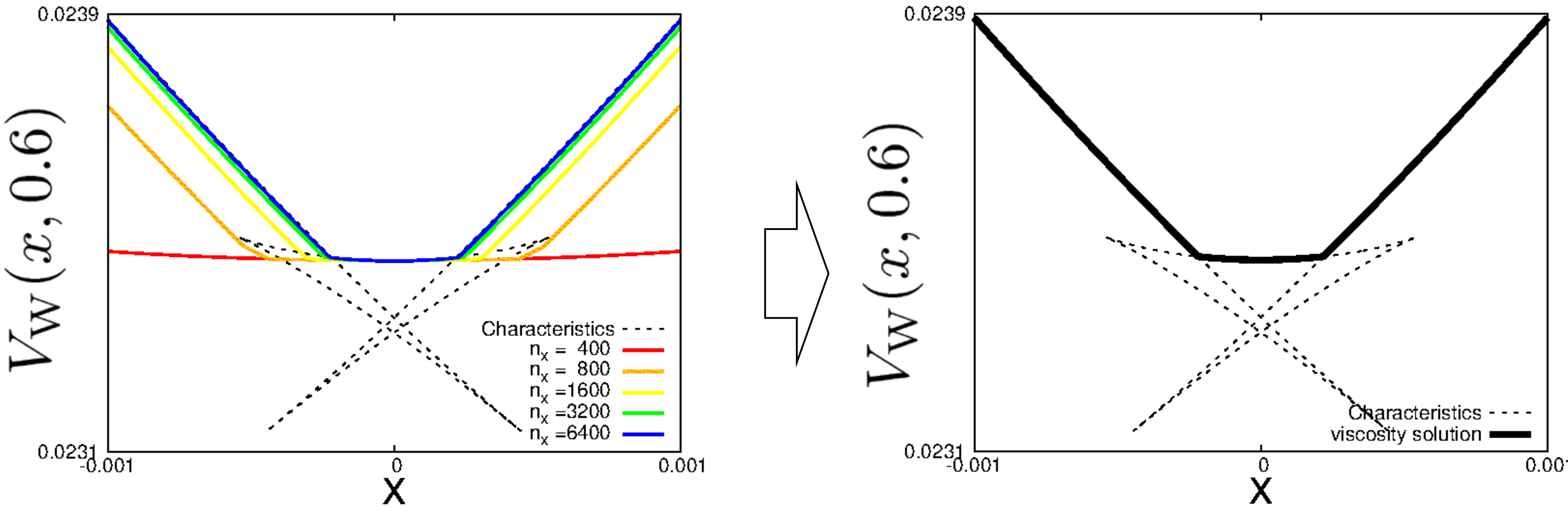
$$S(x_i, t_j) \simeq \min_{k} \Bigg[\underbrace{L\left(\frac{x_i + x_k}{2}, \frac{x_i - x_k}{\Delta t}, \frac{t_j + t_{j-1}}{2} \right) \Delta t}_{\text{Short-time action (line)}} + \underbrace{S(x_k, t_{j-1})}_{\text{Value function at } t_{j-1} \text{ (dot)}} \Bigg]$$



Value function is given in this area.

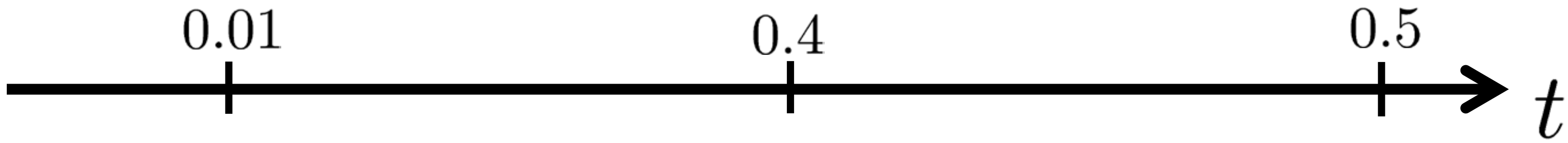
1. Calculate the short-time action and the value function for all lattice points of t_{j-1} respectively.
2. Find the minimum point in the points of t_{j-1} to give us the value function $S(x_i, t_j)$.

3. Results (Wilsonian effective potential, $t = 0.6$)

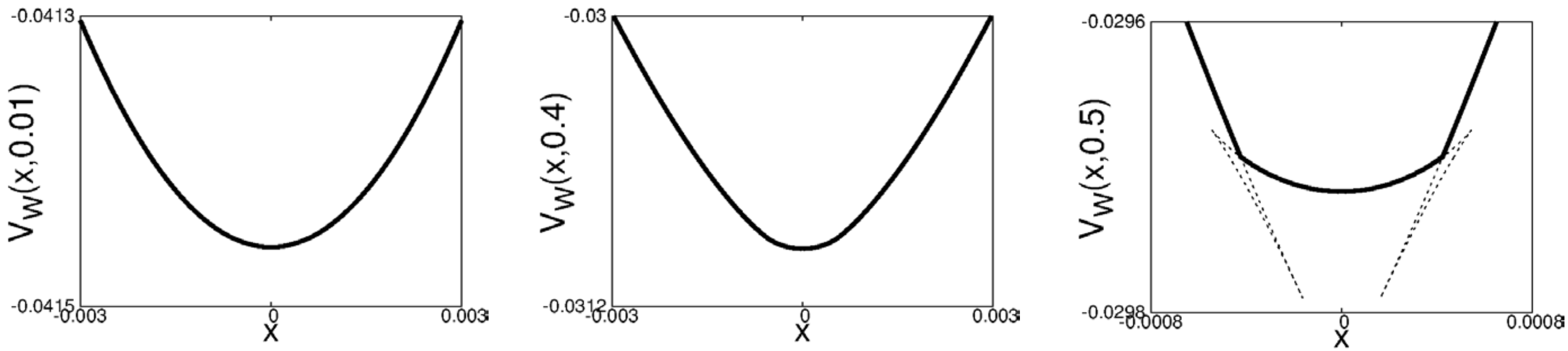


Viscosity solution $V_{\mathrm{W}}(x, t)$ is the continuous and maximum brunch solution in the multi-valued region.

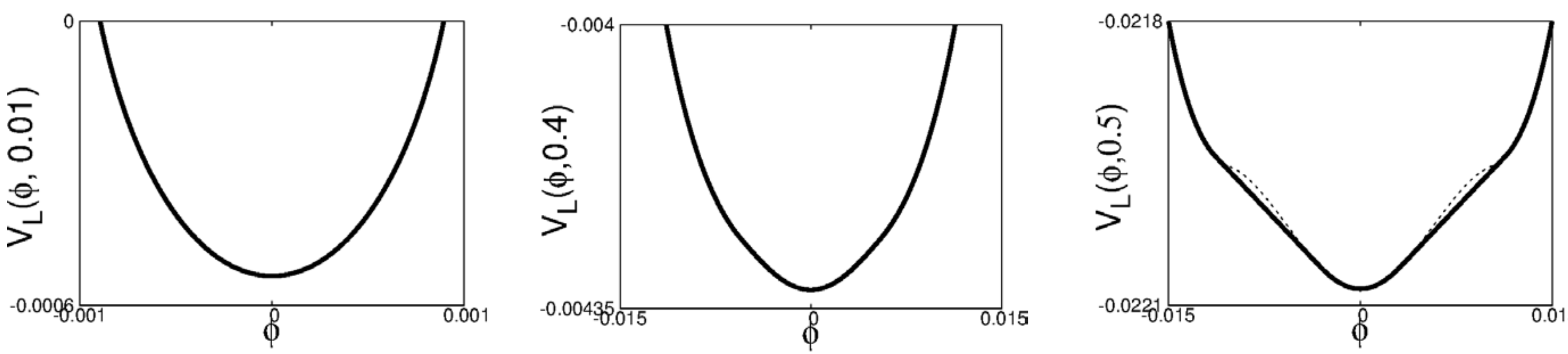
t-evolution of Wilsonian effective potential and Legendre effective potential

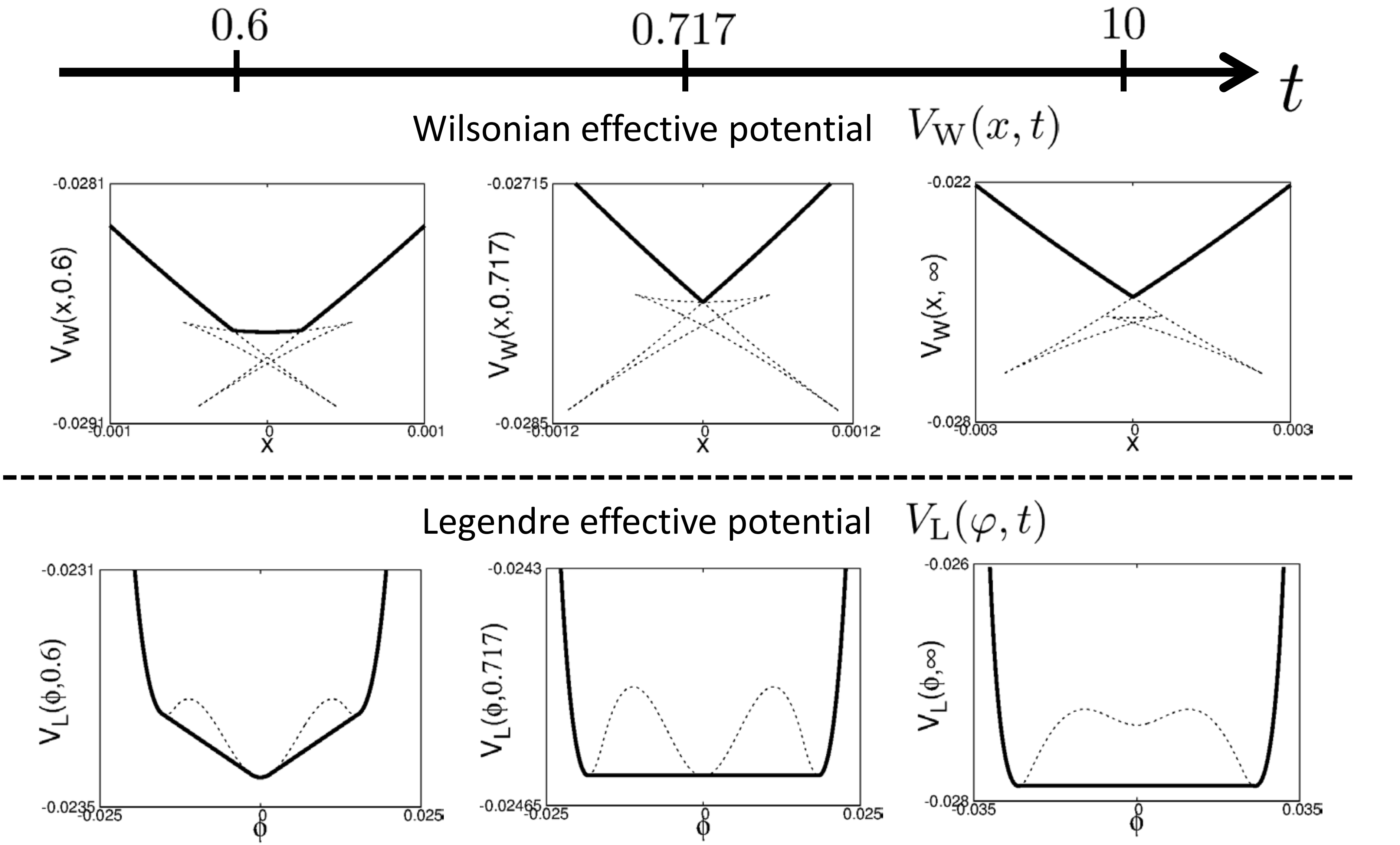


Wilsonian effective potential $V_{\mathrm{W}}(x,t)$



Legendre effective potential $V_{\mathrm{L}}(\varphi,t)$



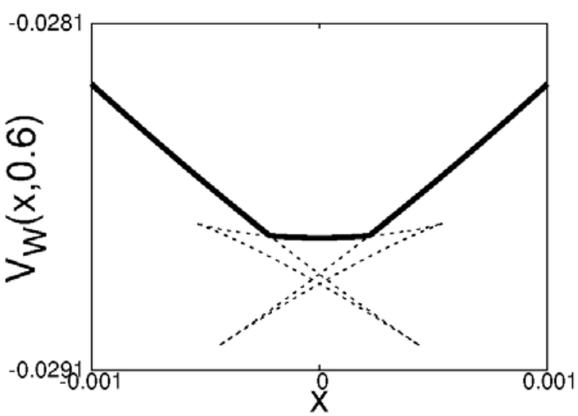


Legendre effective potential (solid line) is a convex envelope of the numerical result of the characteristic equations (broken line).
The convex Legendre effective potential has the global minimum only which corresponds to the vacuum.

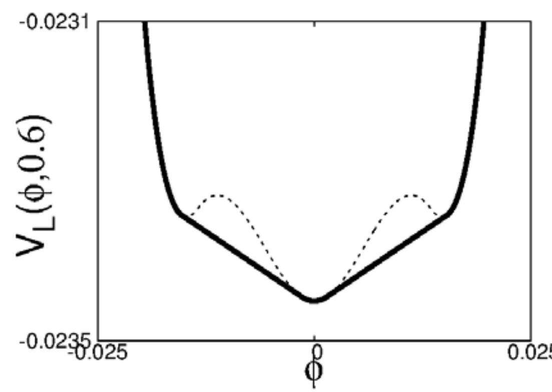
5. Summary and Outlook

- The NPRGE of the NJL model is the H-J type equation and the solution has some singularities.
- The viscosity solution (weak solution) is a single-valued and singular solution.
- The obtained viscosity solution perfectly describes the physically correct vacuum even in the case of the first order phase transition, which is also demonstrated by the auto-convexification of the Legendre effective potential.

Wilsonian effective potential $V_{\mathrm{W}}(x,t)$



Legendre effective potential $V_{\mathrm{L}}(\varphi,t)$



- We are going to apply this method to the finite temperature and density QCD and improve the local potential approximation to the 2nd order PDE.