Scale and Conformal Invariance

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I apologize that the references given here do not do justice to many interesting works. Self references are omitted, too.

If we have a fixed point, it must not have a mass scale, for otherwise, sufficiently many re-scalings of l would change the Hamiltonian.

Suppose we have such a theory without a mass scale. In the simplest case this means that all the correlation functions are power laws. The naive symmetry group:

$$ISO(d) \rtimes \mathbb{R} \ .$$

Surprisingly, we often discover that the symmetry group is actually

$$SO(d + 1, 1)$$

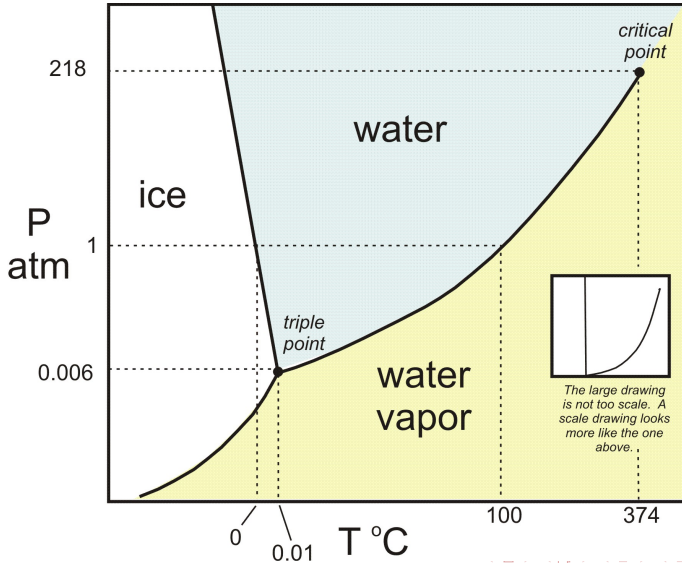So we have d unexpected conserved charges.

Such symmetry enhancement leads to powerful constraints on the spectrum of the theory and its correlation functions. For example, the hydrogen atom is solvable because of a similar symmetry enhancement $SO(3) \rightarrow SO(4)$.

The idea that this symmetry enhancement is a general phenomenon in QFT has been around for many decades (Migdal, Polyakov, Wilson, and others wrote about this already in the 70s).

It has been realized fairly early (although I am not sure when and by whom) that unitarity is a key ingredient in having these d extra generators.

General Comments on Theories without a Mass Scale

Connection to Supersymmetry: The set of examples in $d = 4$ has been rather scarce before the 90s, when Seiberg et al. have solved for the infrared dynamics of many nontrivial examples. All the evidence points to these theories being conformal, and not just scale invariant.

Actually, not all unitarity scale-invariant theories are conformal. A 3d Abelian gauge theory is a counter-example (i.e. QED_3). We will discuss it in detail soon.

This counter-example to scale$\rightarrow$conformal is special for two reasons

- It is free.
- On $\mathbb{R}^3$ all local observables coincide with those of a free scalar in 3d. The latter is conformal. Hence, the non-conformality of a free photon is only a "formality."

There has been a lot of recent work on the problem of scale/conformal invariance in $d = 4$. Let us note that, as in $d = 3$, there is a simple free counter-example: the two-form gauge theory

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu$$

$$\mathcal{S} \sim \int d^4x \left(\partial_{[\mu} B_{\nu\rho]} \right)^2$$

Suppose that

$$T_\mu^\mu = \partial^\nu V_\nu$$

for some **local** operator V_ν. Then the theory is scale invariant and we have the conserved current

$$S_\mu = x^\nu T_{\mu\nu} - V_\mu \ .$$

To prove that a **unitary** scale invariant theory is conformal, one needs to show that

$$T_\mu^\mu = \Box L$$

for some **local** L.

The condition

$$T_\mu^\mu = \Box L$$

might look unfamiliar. However, if it is satisfied we can define

$$T_{\mu\nu}^{NEW} = T_{\mu\nu} - \frac{1}{d-1}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)L$$

Such that $\eta^{\mu\nu}T_{\mu\nu}^{NEW} = 0$. This is called an improvement transformation.

There is a nice argument solving the problem in $d = 2$ [Polchinski, 1988].

$d = 2$ is exceptionally simple because the scaling dimension of L is zero. So we just need to prove that in unitary scale invariant theories

$$T^\mu_\mu = 0 .$$

Strategy: Show that the two-point function $\langle T^\mu_\mu(x) T^\mu_\mu(0) \rangle = 0$ at $x \neq 0$.

Solution for $d = 2$

$$\langle T_{\mu\nu}(q) T_{\rho\sigma}(-q) \rangle = B(q^2) \tilde{q}_\mu \tilde{q}_\nu \tilde{q}_\rho \tilde{q}_\sigma \ ,$$

with $\tilde{q}_\mu = \epsilon_{\mu\nu} q^\nu$. This is the most general decomposition satisfying conservation and permutation symmetry. In a scale invariant theory we must take by dimensional analysis

$$B(q^2) = \frac{1}{q^2} \ .$$

Then,

$$\langle T_\mu^\mu(q) T_\rho^\rho(-q) \rangle \sim q^2 \ .$$

This is a contact term, thus, $T_\mu^\mu = 0$.

There is no hope to repeat an argument of this kind in $d > 2$ because it is not true that unitarity and scale invariance imply that $T^\mu_\mu = 0$. Indeed, in many examples one finds a nontrivial L:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2$$

leads to $T^\mu_\mu = \frac{2-d}{4}\Box(\phi^2)$, i.e. $L = \frac{2-d}{4}\phi^2$.

This is of course a conformal theory and $T^\mu_\mu = 0$ after an improvement.

Counterexample

Take

$$\phi \simeq \phi + c , \qquad \text{for all } c$$

This is consistent because the set of operators where ϕ appears only with derivatives is closed under the OPE. It is local because we have the EM tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 .$$

It is not conformal because the improvement $\sim \left(\partial_\mu\partial_\nu - \partial^2\eta_{\mu\nu}\right)\phi^2$ is not an allowed operator.

In flat space this theory is indistinguishable from the ordinary scalar, it has consistent separated points correlation functions, OPE, consistent anomalies etc.

So this theory is not conformal, but there is no local measurement on $\mathbb{R}^d$ that can distinguish it from a conformal theory.

There are no known scale invariant unitary theories which are distinguishable on $\mathbb{R}^d$ from conformal theories.

In perturbation theory we have a clear list of candidates for L and V_μ and we need to check if the equations $T^\mu_\mu = \partial^\mu V_\mu$, $T^\mu_\mu = \Box L$ are satisfied. This has been checked very explicitly in many 4d models [Grinstein-Fortin-Stergiou] and a beautiful general argument (again in 4d) was offered by [Luty-Polchinski-Rattazzi] as well as [Osborn] and [Grinstein-Fortin-Stergiou].

The problem also simplifies when there is a weakly-coupled holographic dual [Nakayama]. There is some evidence that all unitary solutions to $10d/11d$ Einstein equations with fluxes that are scale invariant are also conformal invariant. If that can be shown in some generality for $d > 2$ that would be fantastic.

Some simplification also takes place in SUSY theories, see for example [Antoniadis-Buican, Zheng, Nakayama, Fortin-Grinstein-Stergiou]

Outline of the Argument

Idea: since we need to prove that $T = \Box L$, let us try to establish the following **necessary** condition

$$\langle VAC | T_\mu^\mu(p_1)....T_\mu^\mu(p_n) | Anything \rangle_{connected} = 0 \ , \qquad p_i^2 = 0 \ ,$$

and see where this takes us. Let us call this the "vanishing theorem." Of course, we assume unitarity – otherwise there are many counter-examples, which do not obey the vanishing theorem. Hence, the vanishing theorem is a nontrivial necessary condition.

Note: [Luty, Polchinski, Rattazzi] estabslihed the case of $n = 2$, i.e.

$$\langle VAC| T_\mu^\mu(p_1) T_\mu^\mu(p_2)|Anything\rangle_{connected} = 0 \ , \qquad p_1^2 = p_2^2 = 0 \ .$$

A Proof of the Vanishing Theorem

We couple any SFT to a background metric. Then we can consider the generating functional $W[g_{\mu\nu}]$. The UV divergences are characterized by

$$\int d^4x \sqrt{g} \left(\Lambda + \mathfrak{a}R + \mathfrak{b}R^2 + \mathfrak{c}W^2 \right) \ ,$$

Consider metrics of the type

$$g_{\mu\nu} = (1 + \Psi)^2 \eta_{\mu\nu}$$

with $\partial^2\Psi = 0$ then neither of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ contribute.

A Proof of the Vanishing Theorem

Thus $W[\Psi]$ is well defined up to a momentum-independent piece.
We define

$$A_n(p_1, ..., p_{2n}) = \frac{\delta^n W[\Psi]}{\delta\Psi(p_1)\delta\Psi(p_2)...\delta\Psi(p_{2n})}$$

and we will choose all the momenta to be null, $p_i^2 = 0$.

A Proof of the Vanishing Theorem

Let us start from $n = 2$. We can prepare forward kinematics $p_3 = -p_1$ and $p_4 = -p_2$. We have the dispersion relation

$$A_4(s) = \frac{1}{\pi} \int ds' \frac{ImA_4(s')}{s - s'} + subtractions \ , \qquad s = (p_1 + p_2)^2 \ .$$

By dimensional analysis, $ImA_4 = \kappa s^2$. We immediately see that $ImA_4 = 0$. Had it not been zero, we would have needed a subtraction which goes like s^2.

A similar argument proceeds for all the amplitudes A_{2n}, in other words, in forward kinematics

$$ImA_{2n} = 0$$

.

A Proof of the Vanishing Theorem

Now we use unitarity, more precisely, the optical theorem.

All the contributions to ImA_4 are positive definite since there is just one cut (s-channel and t-channel, depending on whether $s > 0$ or $s < 0$).

Hence,

$$\langle T^{\mu}_{\mu}(p_1) T^{\mu}_{\mu}(p_2)|Anything \rangle = 0 \ , \qquad p_1^2 = p_2^2 = 0$$

Starting from $n = 3$, the situation is tougher.

- There are many cuts.
- Many of them are generally non-positive.

A Proof of the Vanishing Theorem

However, after some work one can show inductively that the non-positive cuts are absent. Thus,

$$\langle T_\mu^\mu(p_1) T_\mu^\mu(p_2)...T_\mu^\mu(p_n)|Anything\rangle = 0\,, \qquad p_1^2 = p_2^2 = .. = p_n^2 = 0$$

We have thus proved our nontrivial necessary condition.

A Proof of the Vanishing Theorem

The fact that $T_\mu^\mu(p_1) T_\mu^\mu(p_2)...T_\mu^\mu(p_n) = 0$ for all n on the light cone is very suggestive. Indeed, if one could say that this product is analytic in momentum, the vanishing on the light cone would imply the existence of some local L such that $T_\mu^\mu = \Box L$, simply by Taylor expanding around the light cone.

Let us see how to say this precisely:

Consider the effective field theory coupling Ψ (the conformal factor of the metric) to the SFT

$$\mathcal{S} = \int d^4x (\partial \Psi)^2 + \mathcal{S}_{SFT} + \frac{1}{M} \int d^4x \Psi T^\mu_\mu + \cdots$$

where the $\cdots$ are determined by diff invariance.

To leading order in $energy/M$, the S-matrix for Ψ scattering into SFT states is governed by our vanishing correlation functions $\langle T^\mu_\mu(p_1) T^\mu_\mu(p_2)...T^\mu_\mu(p_n)|Anything\rangle = 0$.

A sufficient condition

Clearly, if the SFT is a CFT and $T^\mu_\mu = \Box L$, then the coupling $\frac{1}{M} \int d^4 x \Psi T^\mu_\mu = \frac{1}{M} \int d^4 x \Box \Psi L$ vanishes on-shell and can be removed by a local change of variables, consistent with the trivial S-matrix.

But we can also argue for the converse: a trivial S-matrix means the theories are decoupled. Hence, there is a local L such that $T^\mu_\mu = \Box L$.

An S-matrix Digression

Let us take two theories A and B. Suppose there is a local change of variables connecting A and B. Then $S_A = S_B$.

Does it follow from $H_A \simeq H_B$ and $S_A \simeq S_B$ that there is a local change of variables connecting A and B? The answer is negative. For example, the kink-field duality, electric-magnetic duality...

However, here we just have a *small perturbation* of an existing model with trivial S-matrix. If such a small perturbation does not affect the S-matrix, then the perturbation must vanish on-shell and the change of variables needs to be local.

It is like saying that the S-matrix characterizes the physical theory modulo topological degrees of freedom that don't play any role in $\mathbb{R}^4$.

A sufficient condition

Let us explain how the 2-form fits into this. We cannot solve $T_\mu^\mu = \Box L$. However, since the theory is physically indistinguishable in $\mathbb{R}^4$ from a conformal theory, the S-matrix is insensitive to this subtle zero mode that is absent. So the vanishing theorem is obeyed.

Conclusion

Our precise conclusion is that unitary scale invariant theories are either conformal or indistinguishable from conformal theories on $\mathbb{R}^4$. This means that, for all practical purposes, scale invariance and unitarity imply conformality.

A List of a Few Tangible Challenges

- Perturbative proof that in $d = 3$ scale invariance implies conformal invariance in CS+matter theories.
- A connection between scale/conformal invariance and Entanglement Entropy.
- A holographic understanding of why scale invariance implies conformal invariance.