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Domain Wall Renormalization Approach for 2d Ising spin model

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+ Motivation

- What is "Domain Wall Renormalization Group" (DWRG)?
  - How to define coarse graining of domain walls?
  - Domain wall representation corresponds to "loop" dynamics.
  - A sort of tensor network renormalization group (TRG) method
    We will clarify detailed structures of the TRG transformation.

- The 2d Ising model
  - A best work bench for non-perturbative renormalization group approach
  - Non-trivial magnetization
  - Long history and numerous approaches
    Ex. Onsager's exact solution,
    Various RG approaches already proposed

- Extension to contain external magnetic field
  - Oriented Domain Wall representation
  - High Temperature representation

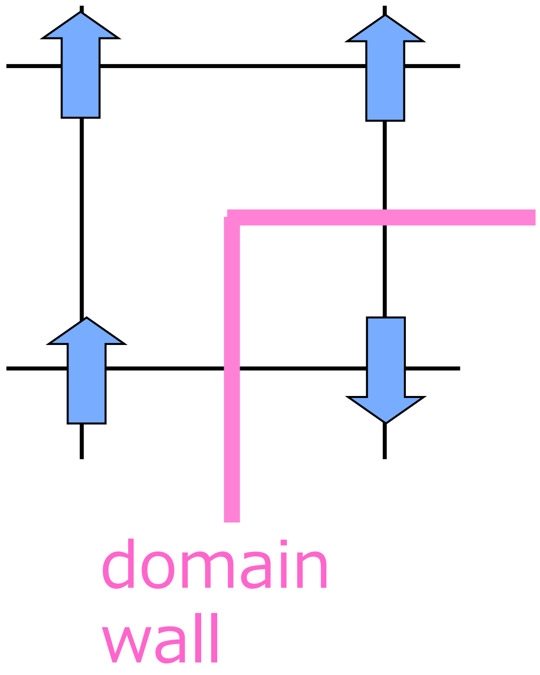
References

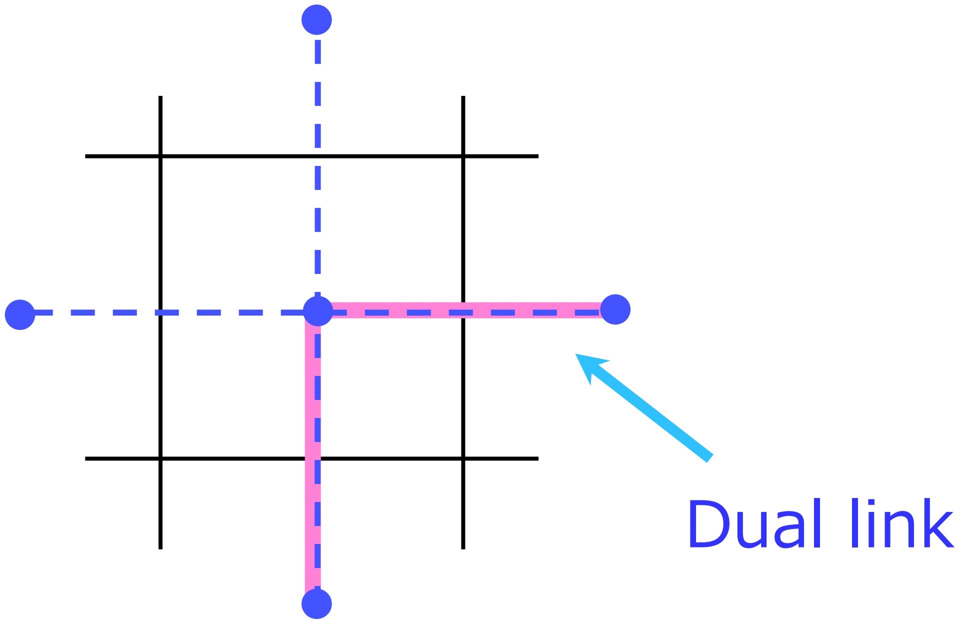
- Triangle lattice
  - M. Levin and C. P. Nave, Phys. Rev. Lett. 99, 20601 (2007)
- Square lattice
  - Z.-C. Gu, M. Levin and X. G. Wen, Phys. Rev. B78, 205116 (2008), arXiv:0806.3509 [cond-mat.str-el]
  - M. Hinczewski and N. Berker, Phys. Rev. E77, 011104 (2008)
- Our works
  - Aoki, Kobayashi & Tomita, JPS2008
  - Aoki, Kobayashi & Tomita, IJMPB, 23 (2009) 3739.
  - Aoki, Fujii, Kobayashi & Sato

    RIMS Kokyuroku No.1904 (2014) 13-30.
  - Aoki, Fujii,Kobayashi,Sato & Yoshimura, JPS2014

+Definition of domain wall

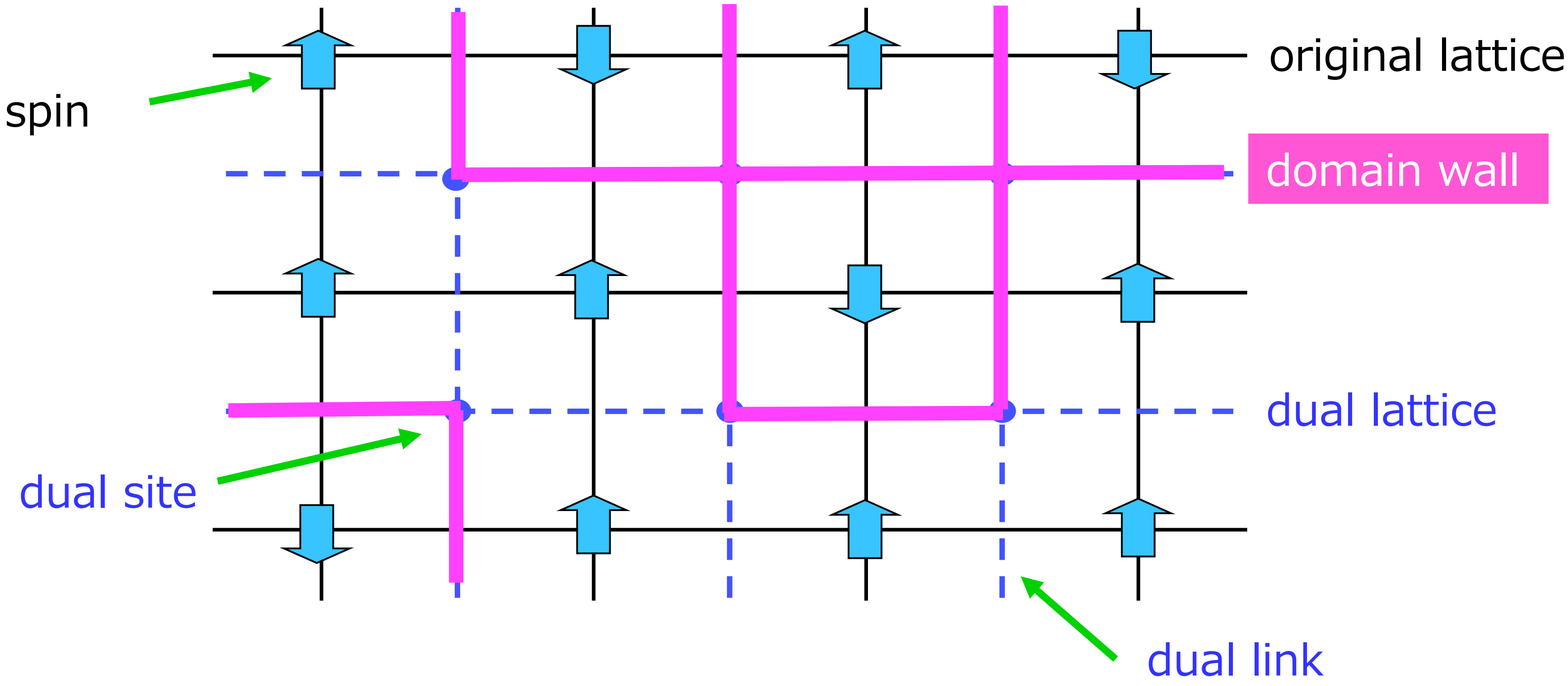
Domain walls are the boundary of up and down spin.





Domain walls live on the dual links.

+ Domain wall representation of spin configuration



Domain walls constitute the boundaries of up and down spins.
Domain walls make "loops".

+ How to represent statistical weights in terms of domain walls

Spin variables $\sigma_i \ (= \pm 1)$

Domain wall variables $\alpha_{ij} \ (= \sigma_i \sigma_j)$ *2 to 1 mapping

- - - - no domain wall $\alpha = +1$

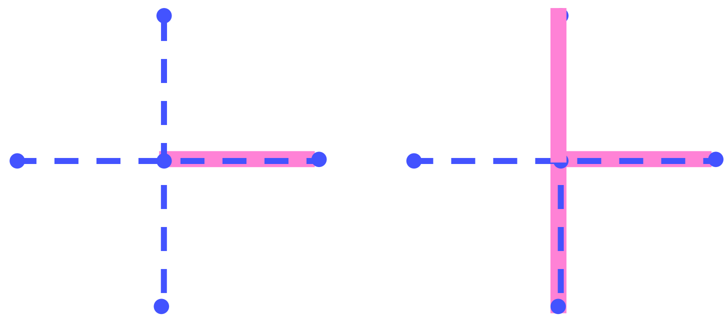
domain wall $\alpha = -1$

Domain walls constitute the boundary of spin up/down domains

⇩

Domain wall is conservative

topological "*loop*" objects



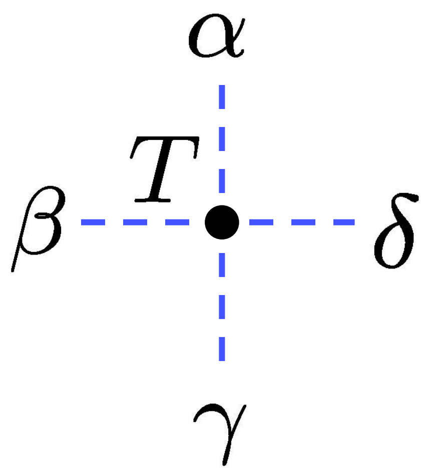
No such configuration

+ Definition of a Tensor

To respect conservation law, we define statistical weights directly.

A vertex "*tensor*"

$$T_{\alpha\beta\gamma\delta}$$



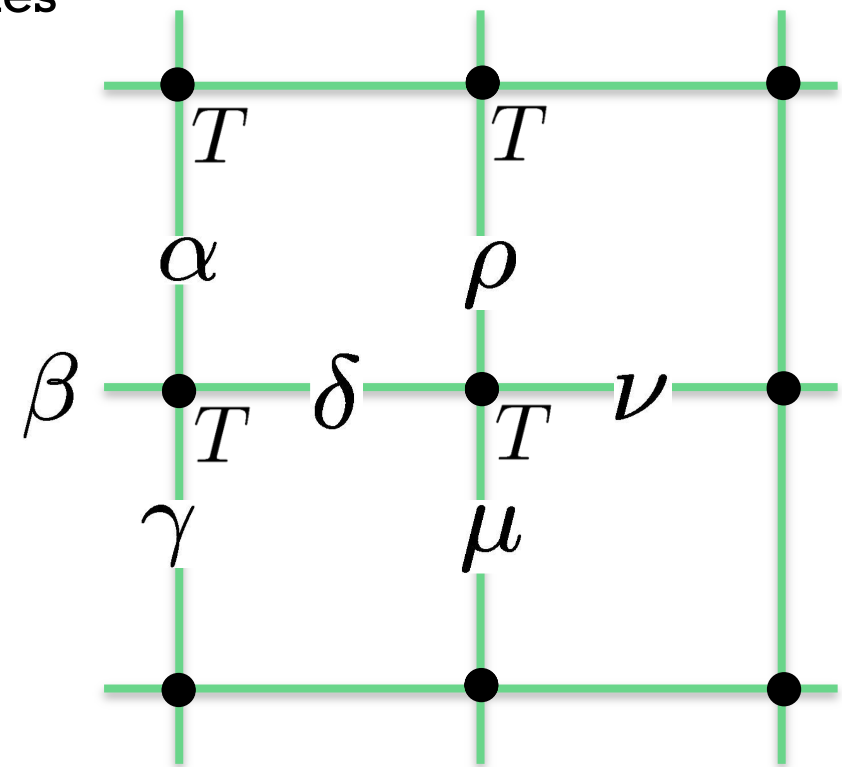
is a function of four domain wall variables around the dual site.

Some elements vanish for vertices breaking the conservation law.
Eventually, non vanishing elements of tensor are only 8 elements.

Partition function

Configuration sum
= Sum of all dual link domain wall variables

$$Z = 2 \sum_{\alpha\beta\delta\gamma\cdots} T_{\alpha\beta\delta\gamma} T_{\delta\mu\nu\rho} \cdots$$



Total products of all tensors
on the dual sites

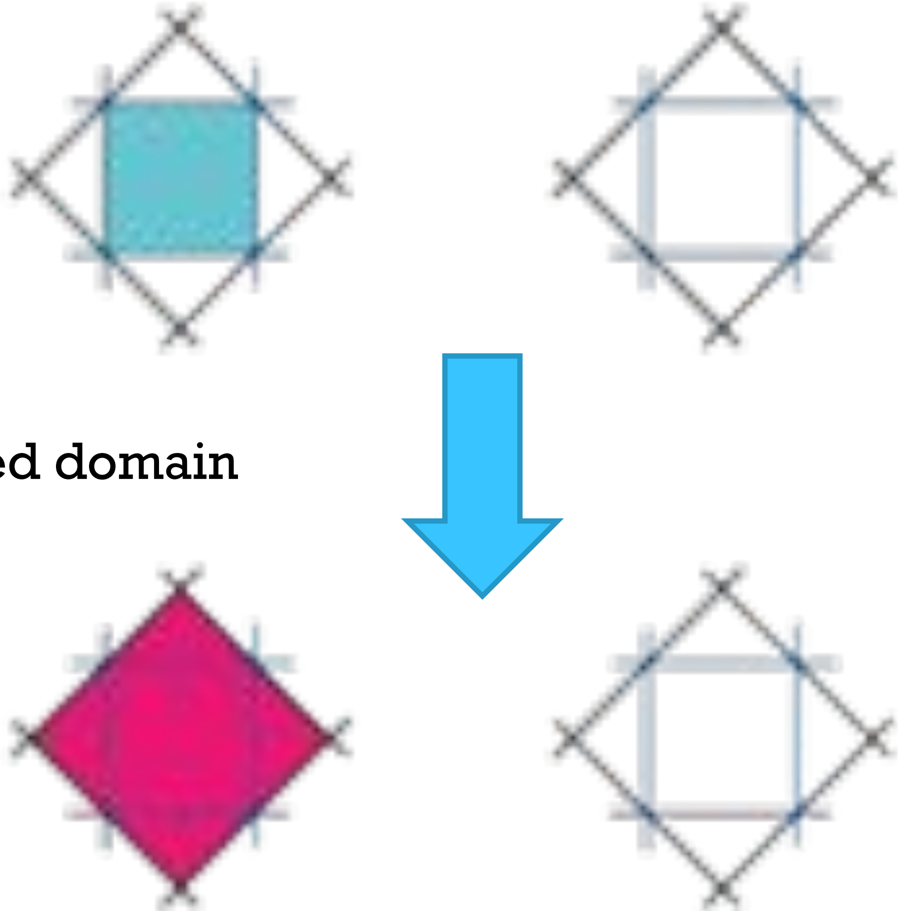
Can be seen as a "*tensor network RG model*"

+ Coarse grained domains

Micro domain

Coarse grained domain

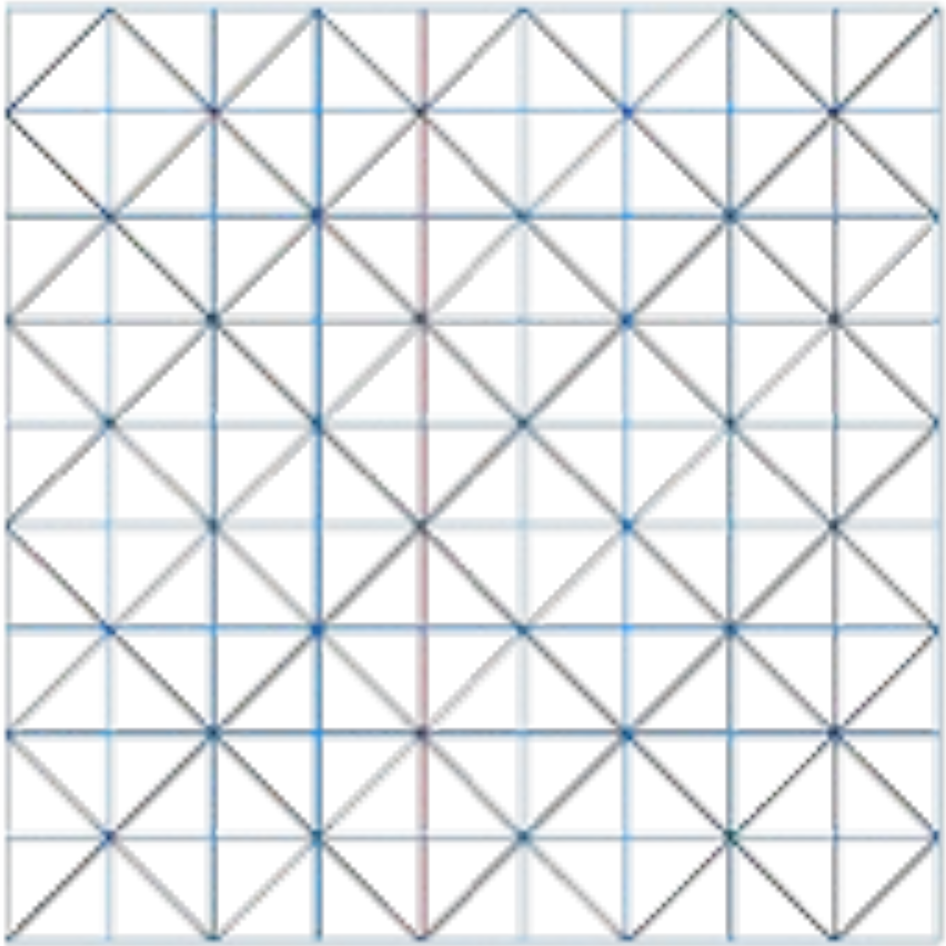
Macro domain



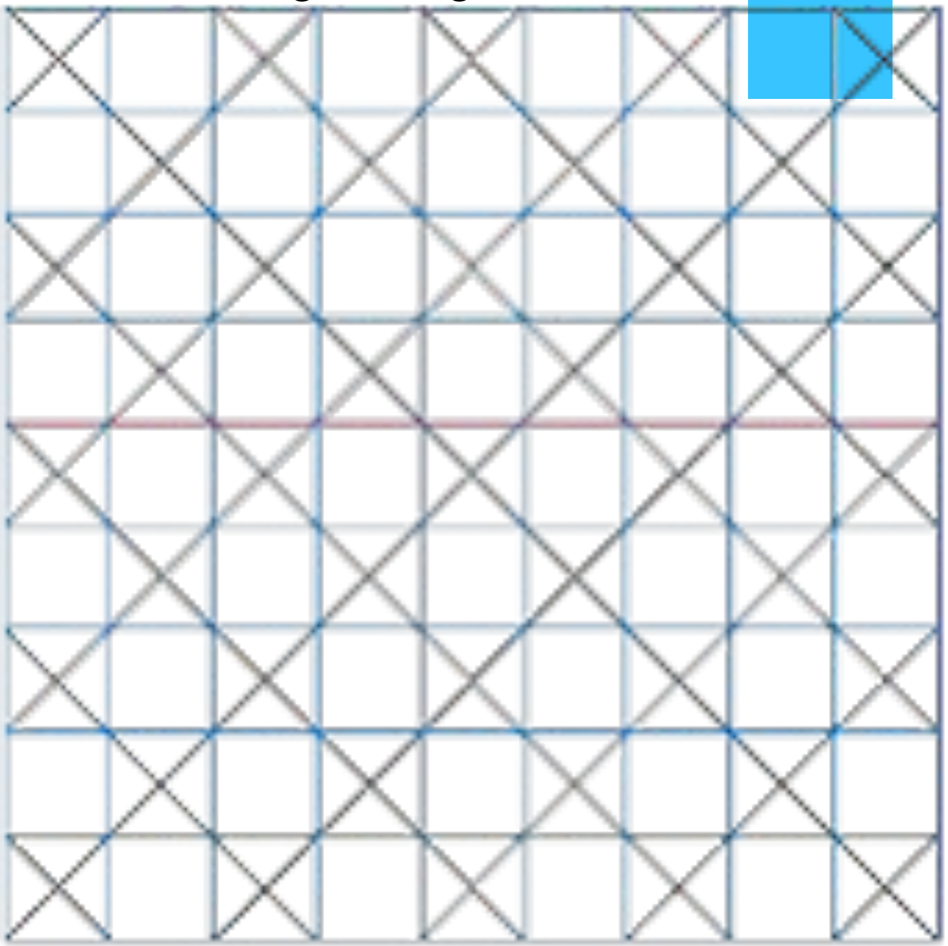
*Domain decimation is equivalent to the spin decimation.

Coarse graining lattice & dual lattice

Coarse graining lattice



Coarse graining dual lattice

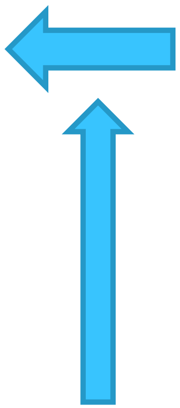


Decimating sites

Decimating domains

+ Local mapping rules (a function)

Coarse grained domain walls (macro variables) should be defined by a "**local**" function of original domain walls (micro variables) in order to satisfy the mapping of domains (RG policy).



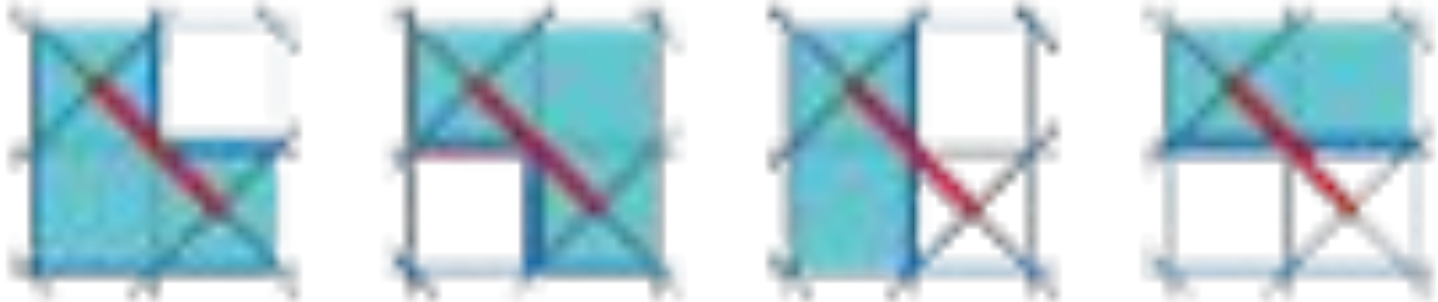
1 macro domain wall

4 micro domain walls

Deterministic function, respecting conservation law

Macro domain walls & conservation law

4 to 1 reduction mapping (in this local domain wall)



Macro domain walls are defined as the boundary of coarse grained domains, thus, they must be conserved.

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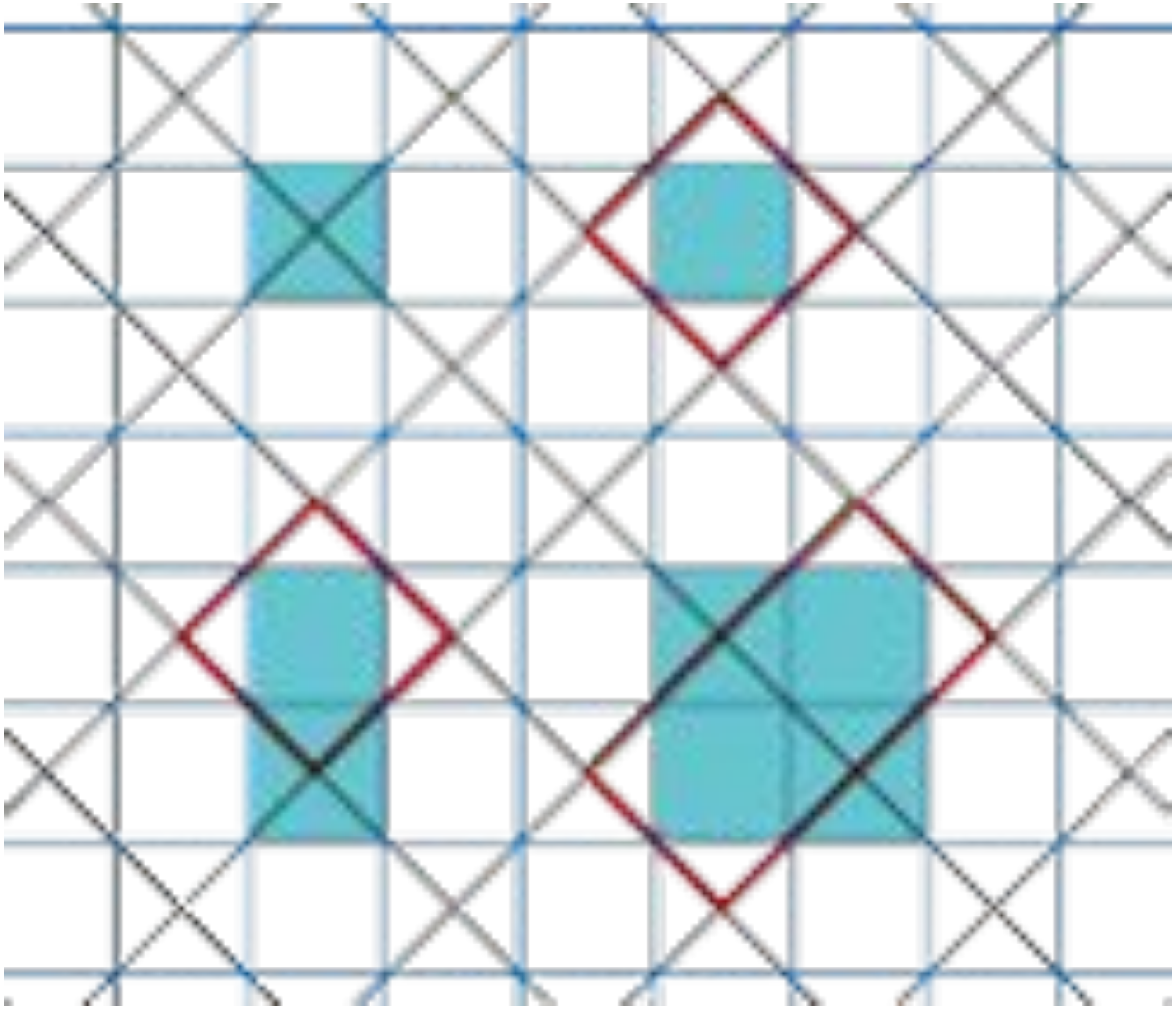
Coarse graining, examples1

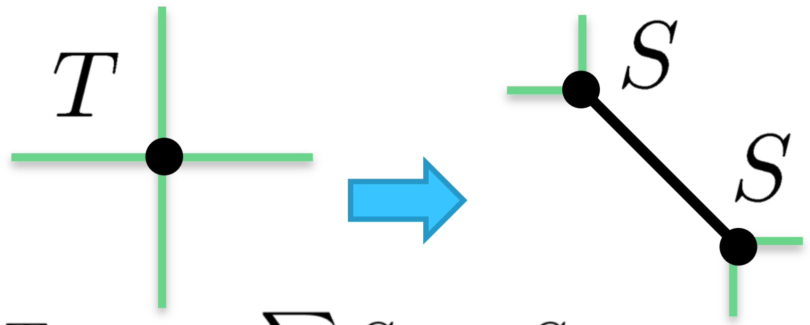


+ How to get macro domain wall variables

Renormalization transformation

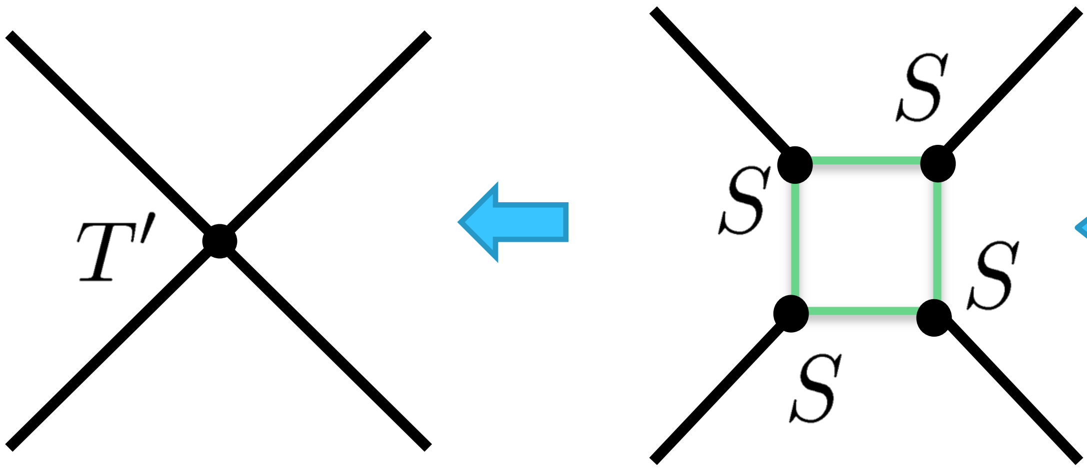
$$T, a$$

$$T', \sqrt{2}a$$



$$T_{\alpha\beta\gamma\delta} \simeq \sum_{M} S_{\alpha\beta M} S_{\gamma\delta M}$$

Optimization problem: **singular value decomposition**

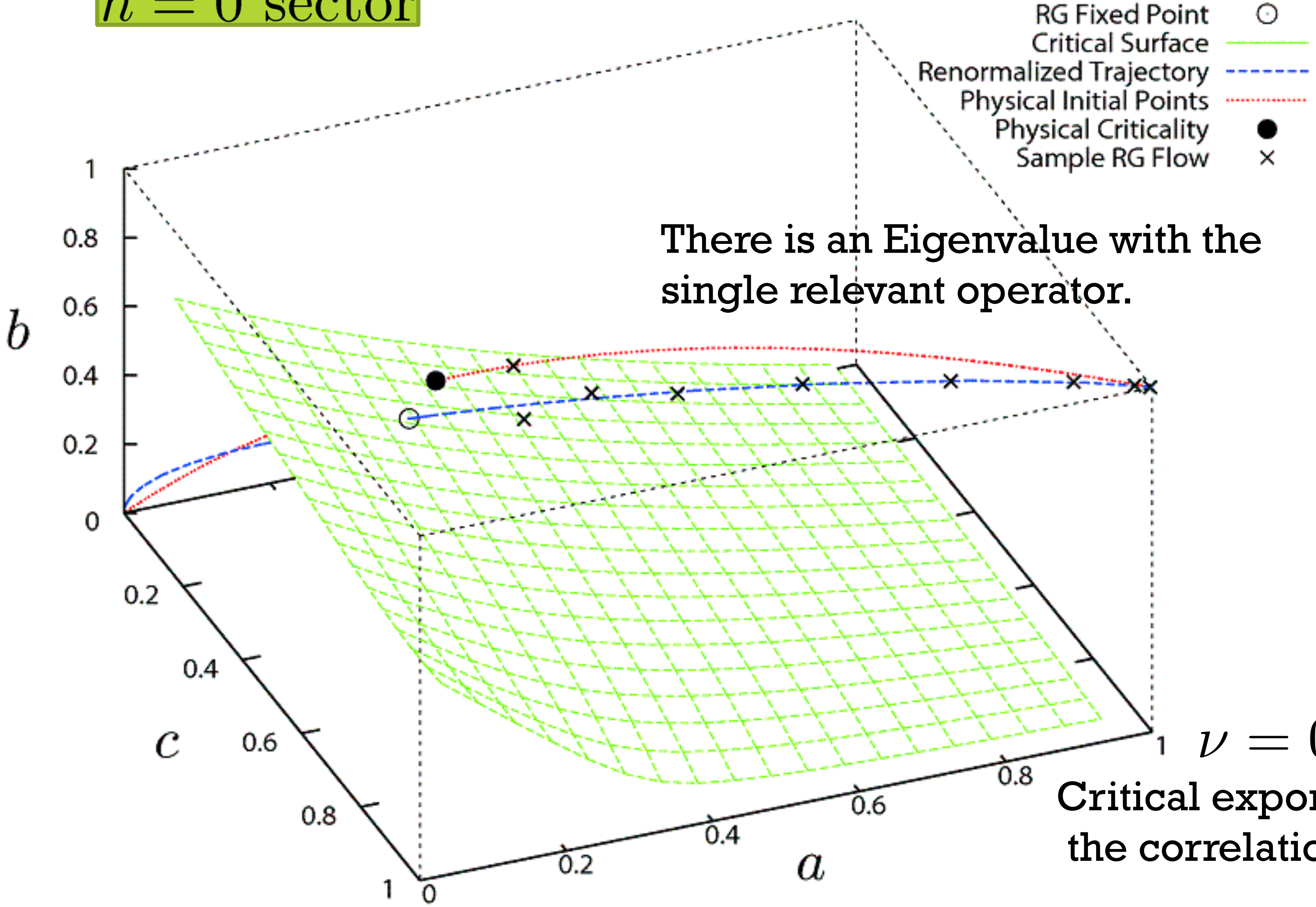


$$T'_{MNKL} = \sum_{\alpha\beta\gamma\delta} S_{\alpha\beta M} S_{\beta\gamma N} S_{\gamma\delta K} S_{\delta\alpha L}$$

Integrating out micro variables

Global view of the RG structure

$h = 0$ sector

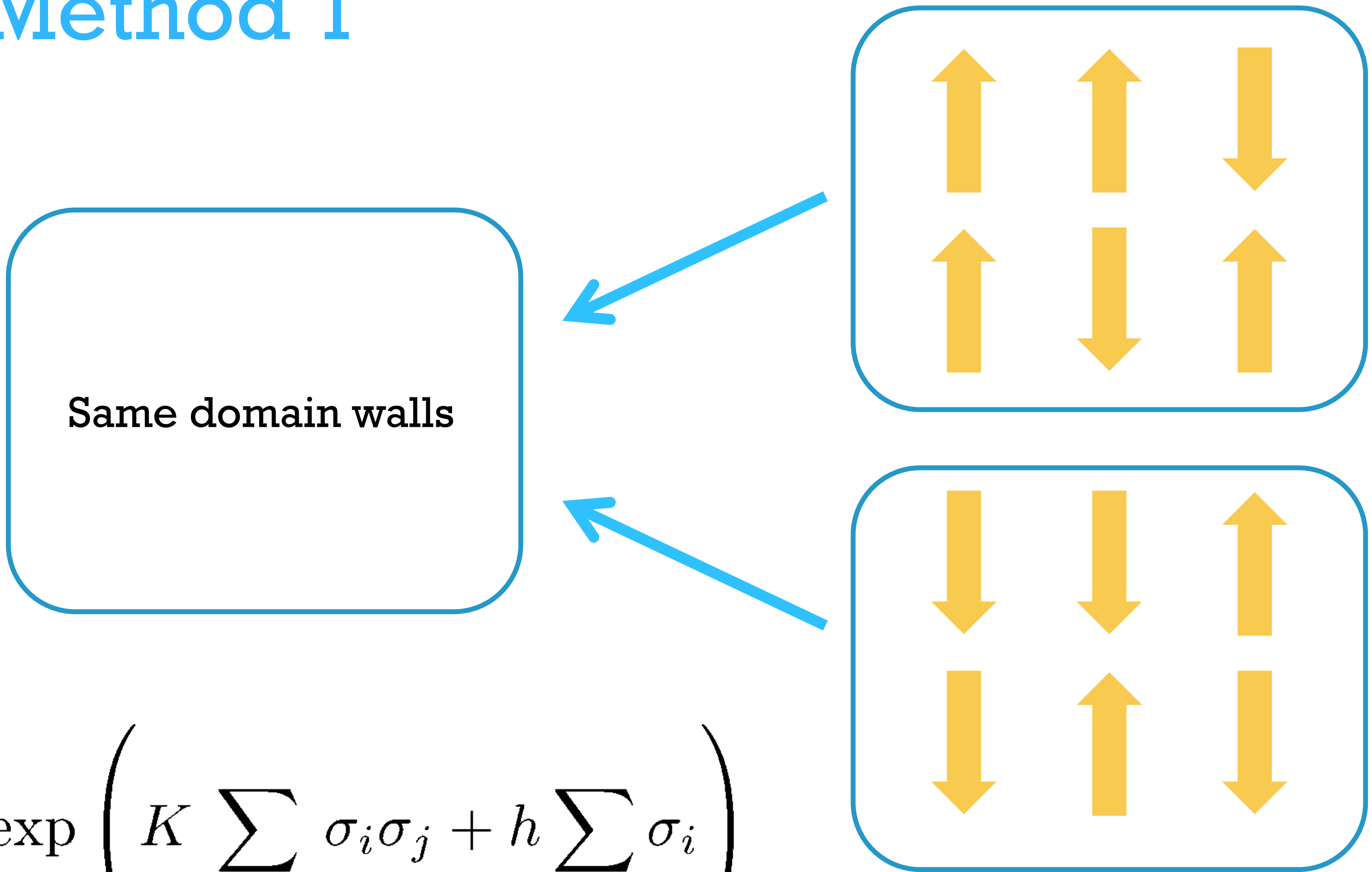


There is an Eigenvalue with the single relevant operator.

$\nu = 0.984$

Critical exponent of the correlation length.

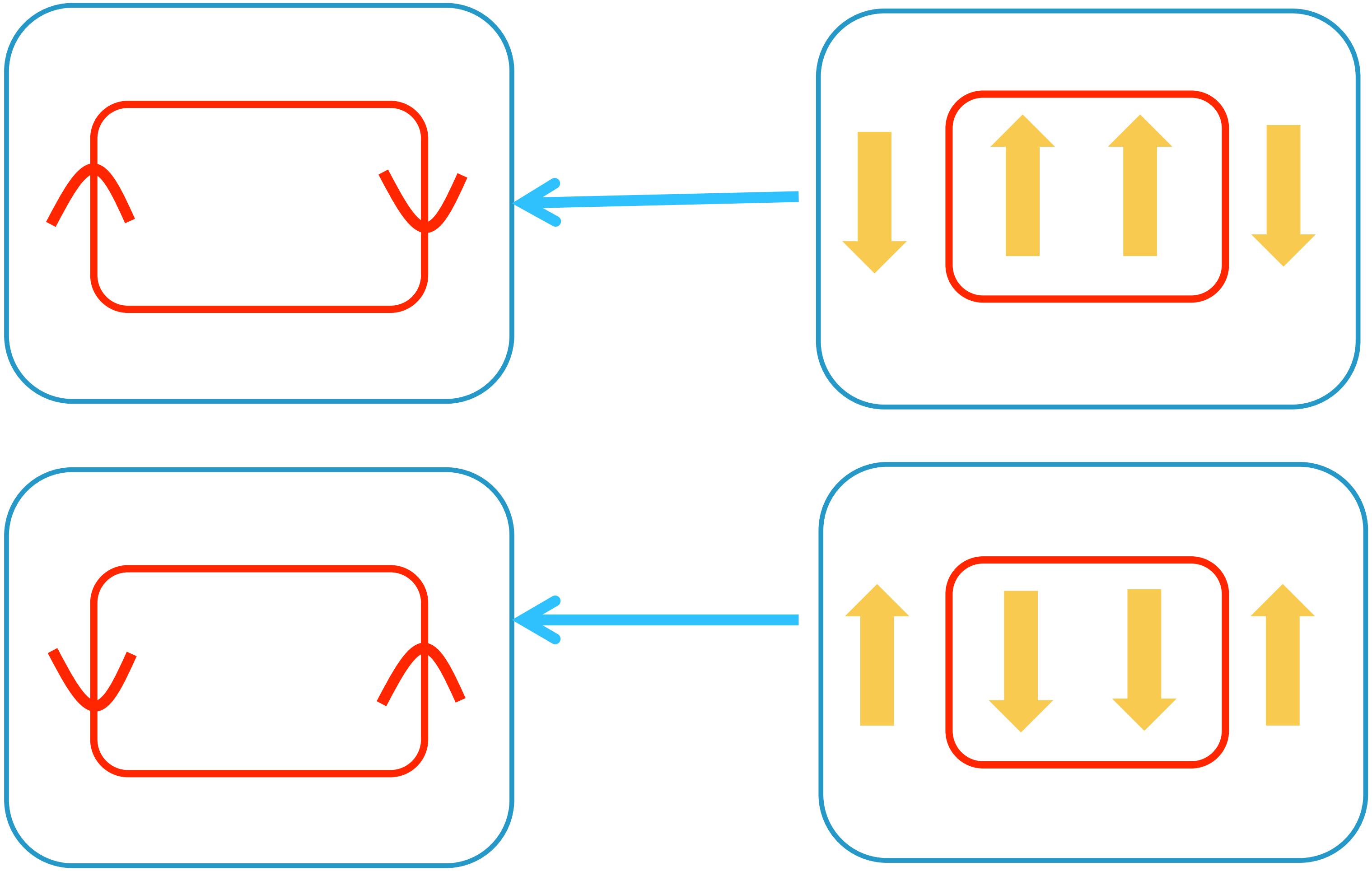
How to introduce magnetic field? Method 1



$$\exp\left(K\sum_{<ij>}\sigma_i\sigma_j + h\sum_i \sigma_i\right)$$

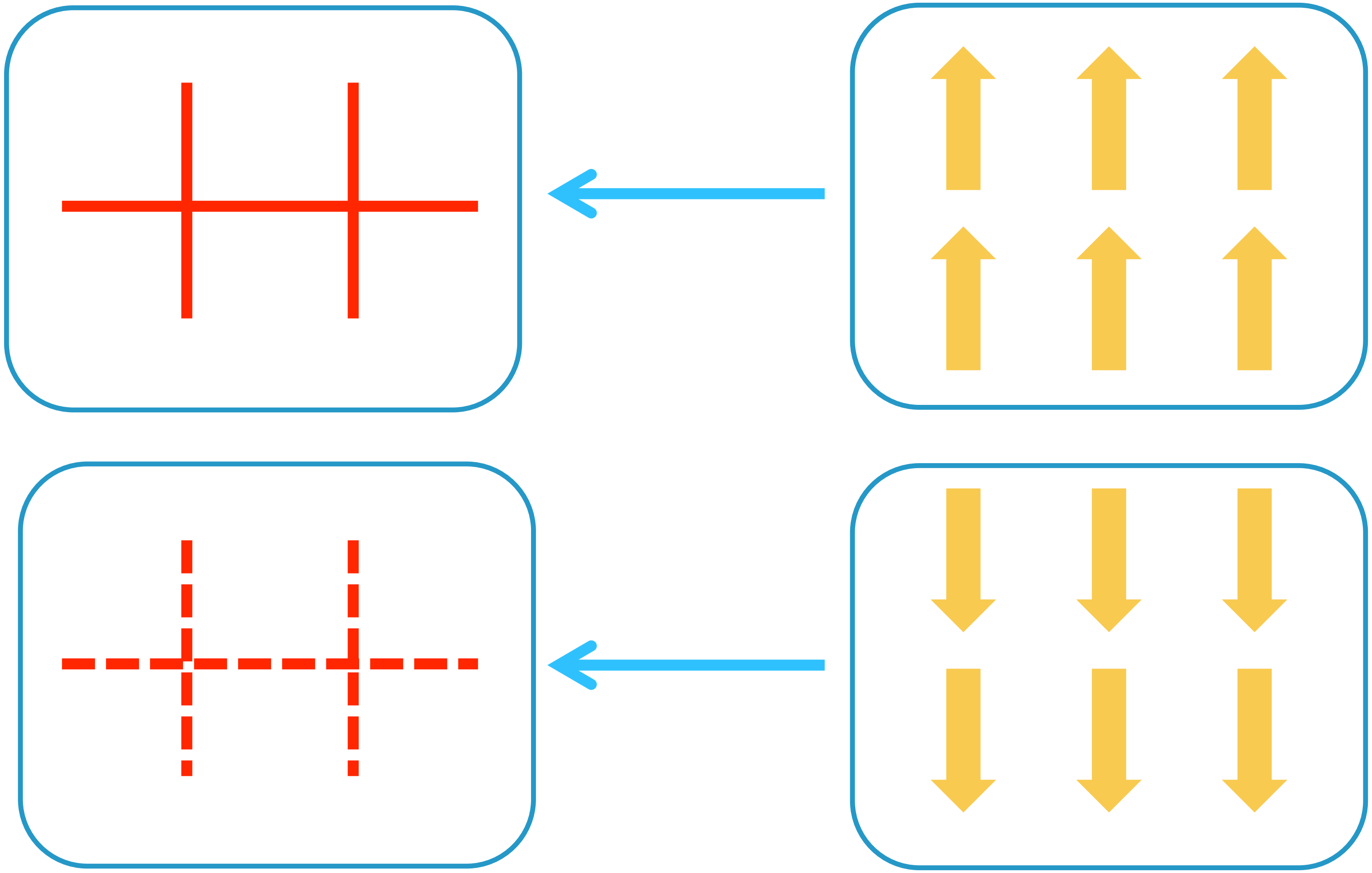
$< ij >$: nearest neighbor pair

Oriented Domain Wall representation



Oriented Domain wall

Up domain & Down domain



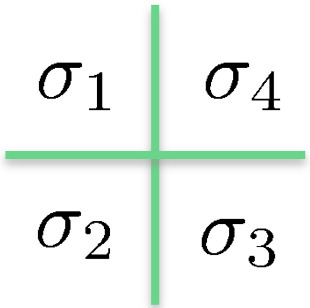
No domain wall

6 dimensional space

$\{N,\ \overline{N},\ b,\ \bar{b},\ a,\ c\}$: Oriented Domain Wall Representation

1 to 1 correspondence

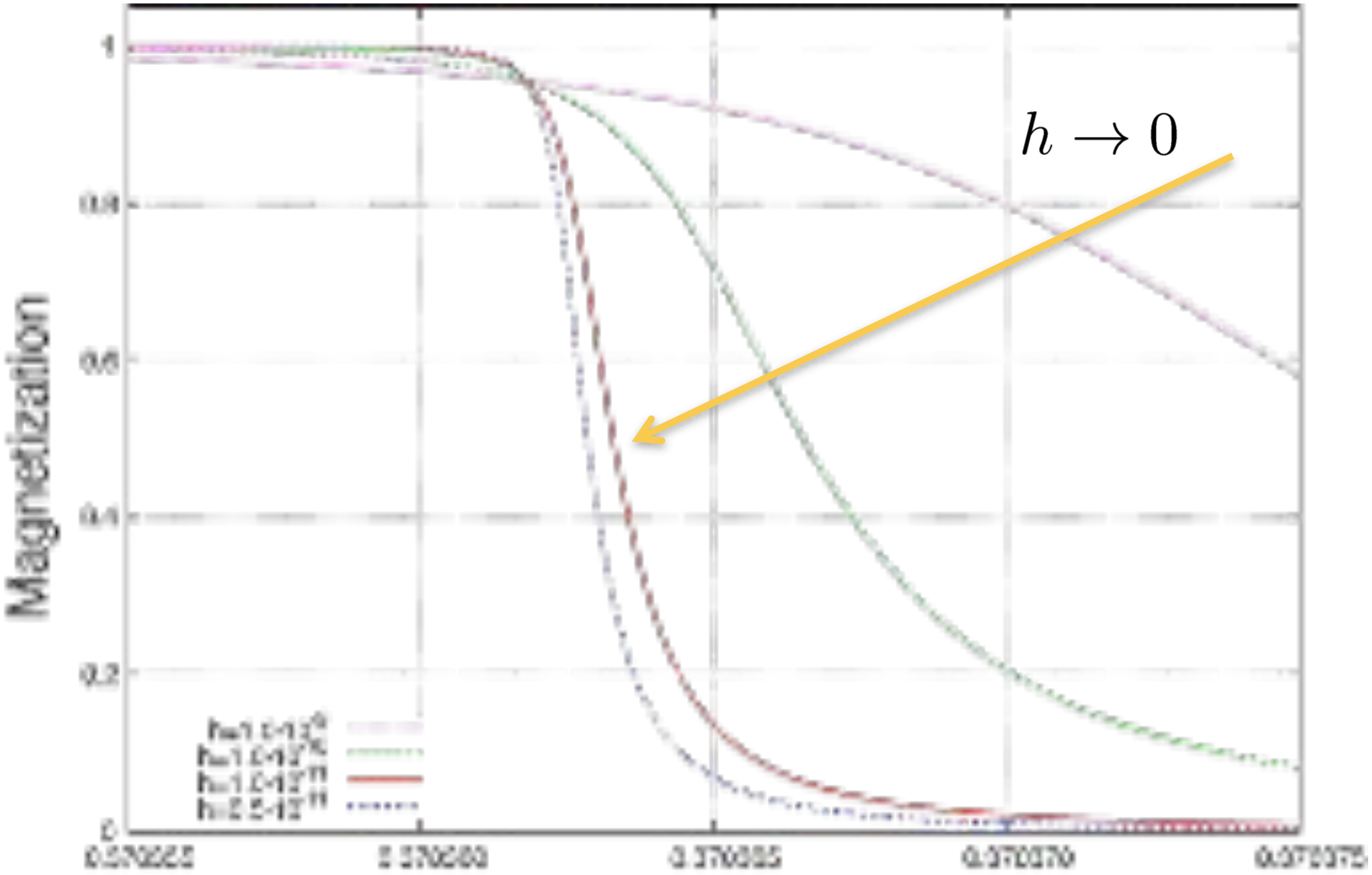
$\{C_0,\ K,\ K_D,\ K_4,\ h,\ h_3\}$: Spin Representation



$$\begin{aligned}\exp\big[C_0 + K(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1) \\ + K_D(\sigma_1\sigma_3 + \sigma_2\sigma_4) + K_4(\sigma_1\sigma_2\sigma_3\sigma_4) \\ + h(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) \\ + h_3(\sigma_1\sigma_2\sigma_3 + \sigma_2\sigma_3\sigma_4 + \sigma_3\sigma_4\sigma_1 + \sigma_4\sigma_1\sigma_2)\big]\end{aligned}$$

+ Magnetization Property

$h \neq 0$ sector



$$\alpha \equiv e^{-2K} \sim T$$

Eigenvalues of Z_2 odd sector around the non-trivial fixed point

2.02141, 1.05465

↓ ?

Critical exponent of susceptibility

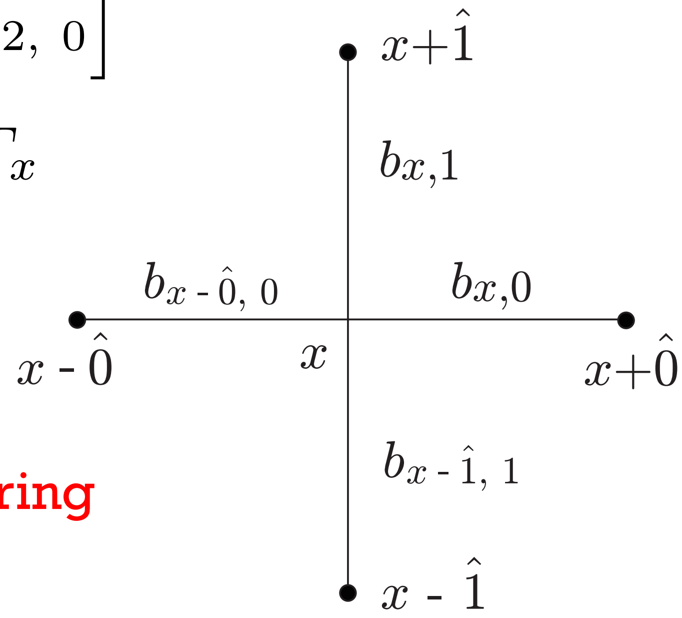
$$\gamma_{\mathrm{DW}} = 2.0276 \quad \Longleftrightarrow \quad \gamma_{\mathrm{exact}} = \frac{7}{4} = 1.75$$

How to introduce magnetic field? Method 2

Bond value representation

$$Z = \sum_{\{\sigma\}} e^{K\sum_{x,\hat{\nu}} \sigma_x \sigma_{x+\hat{\nu}} + h \sum_x \sigma_x}$$

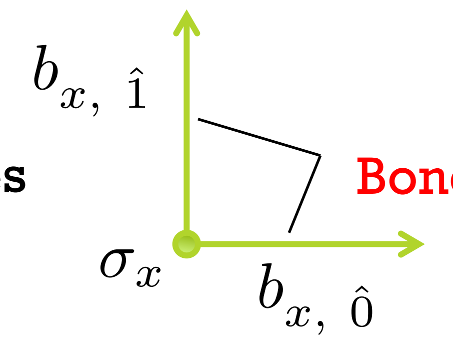
$$\begin{aligned} Z &= 2^V \cosh^{DV} K \cosh^V h \sum_{b_{x,\nu}=\{0,1\}} \sum_{S_x=\{0,1\}} \prod_x \left[\left(\prod_\nu \tanh^{b_{x,\nu}} K \right) \right. \\ &\quad \left. \times \tanh^{S_x} h \, \delta_{\sum_\mu (b_{x,\mu}+b_{x-\hat{\mu},\mu})+S_x \mod 2,\ 0} \right] \\ &= 2^V \cosh^{DV} K \cosh^V h \sum_{b_{x,\nu}=\{0,1\}} \prod_x T_x \end{aligned}$$



$b_{x,\hat{\nu}}$: Bond value

V : Numbers of lattice sites

$D = 2$



Bond string

Bond string rep. with magnetic field

A tensor (bond string rep.)

$$T = \begin{array}{c} \\ 00 \\ 11 \\ 01 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 11 & 01 & 10 \end{array} \\ \begin{pmatrix} 1 & w & f\sqrt{w} & f\sqrt{w} \\ w & w^2 & f\sqrt{w^3} & f\sqrt{w^3} \\ f\sqrt{w} & f\sqrt{w^3} & w & w \\ f\sqrt{w} & f\sqrt{w^3} & w & w \end{pmatrix} \end{array} \quad \begin{array}{l} w = \tanh K \\ f = \tanh h \end{array}$$

Eigen value decomposition

We set infinitesimal magnetic field and calculate by perturbation the eigenvalues of RG transformation around the fixed point annalistically.

New Eigenvalue for Z2 odd sector is found to be $\lambda_h = 1.828$

Critical exponent of susceptibility (preliminary)

$$\gamma_{\text{bs}} = 1.458 \quad \Longleftrightarrow \quad \gamma_{\text{exact}} = \frac{7}{4} = 1.75$$

Summary

- We set up the renormalization group for the Domain Wall representation of the 2D Ising model. The key issue is how to coarse grain the domain walls so that its conservation feature is maintained in the renormalization procedure. We define the coarse grained domain walls by referring to the Tensor Network Renormalization Group technique.

- The Domain Wall RG is extended to include external magnetic field in two ways: Oriented Domain Wall representation and Bond String representation.

- The magnetic quantities are calculated in these two methods, and the critical index obtained seems fair in both methods. We don't understand why yet, and will make further analysis.