ASYMPTOTIC FREEDOM IN HORAVA-LIFSHITZ GRAVITY

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FOM

Based on: G.D., F. Saueressig, M. Schutten, arXiv:1406.4366
and G.D., Frank Saueressig, in preparation

OVERVIEW

FP Structure of Quantum Gravity

Theory Space: Quantum Einstein Gravity
Symmetry: Diffeomorphisms

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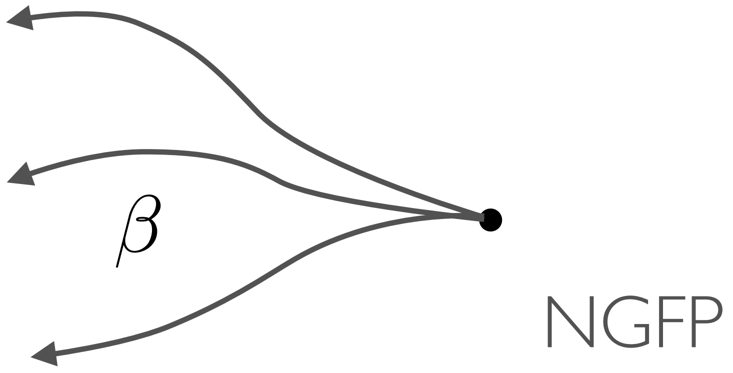

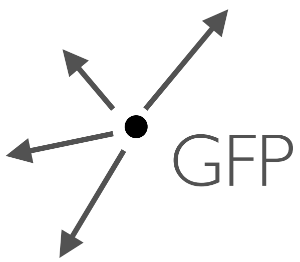

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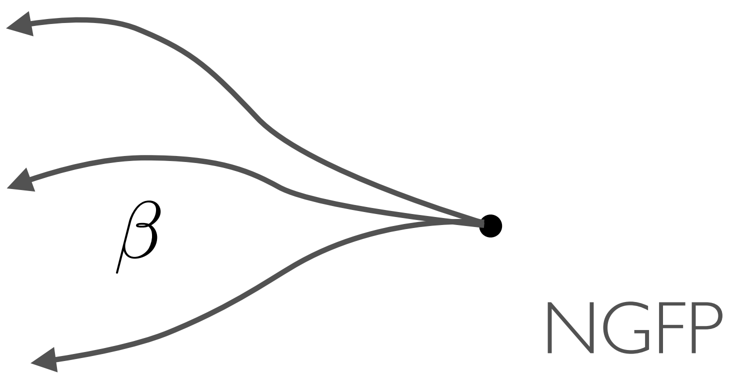

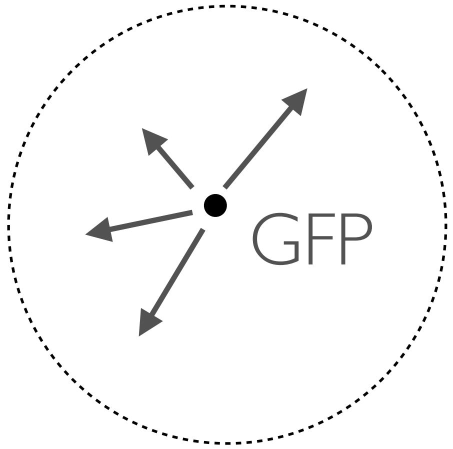

- Critical properties "easy" to determine with perturbative methods

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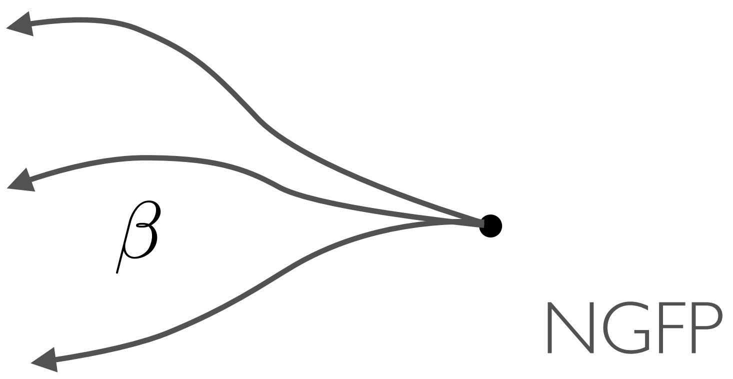

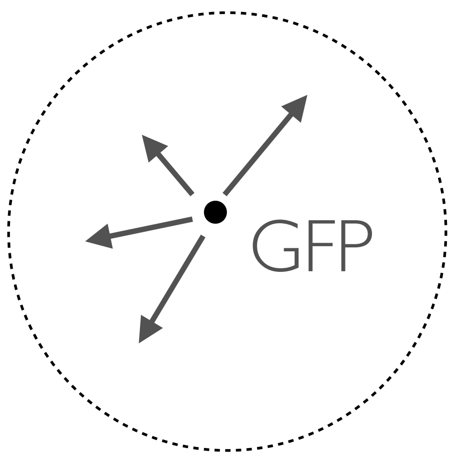

- Critical properties "easy" to determine with perturbative methods
- Unfortunately gravity is perturbatively nonrenormalizable

$$[G_N] = -2$$

Horava-Lifshitz Gravity in a nutshell

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- Anisotropic field theories: change dispersion relation

$$S = \int \left\{ \dot{\phi}^2 - \phi \Delta^z \phi + \sum_{n=1}^{N} g_n \phi^n \right\} dt\, d^d x \qquad \begin{aligned} t &\to b\, t, \\ x &\to b^{1/z}\, x \end{aligned}$$

- Decreases degree of divergence in loop integrals

Horava-Lifshitz Gravity in a nutshell

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- Gravity: Folation-Preserving Diffeomorphisms

$$t \to f(t), \quad \mathbf{x} \to \zeta(t, \mathbf{x})$$

- Natural formulation in ADM variables:

$$ds^2 = N^2 dt^2 + \sigma_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right)$$

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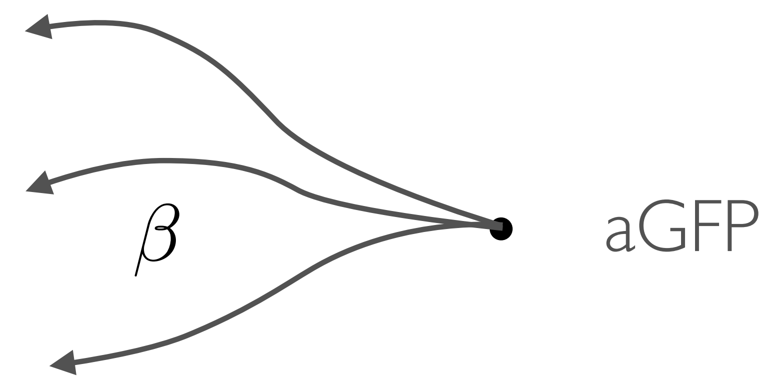
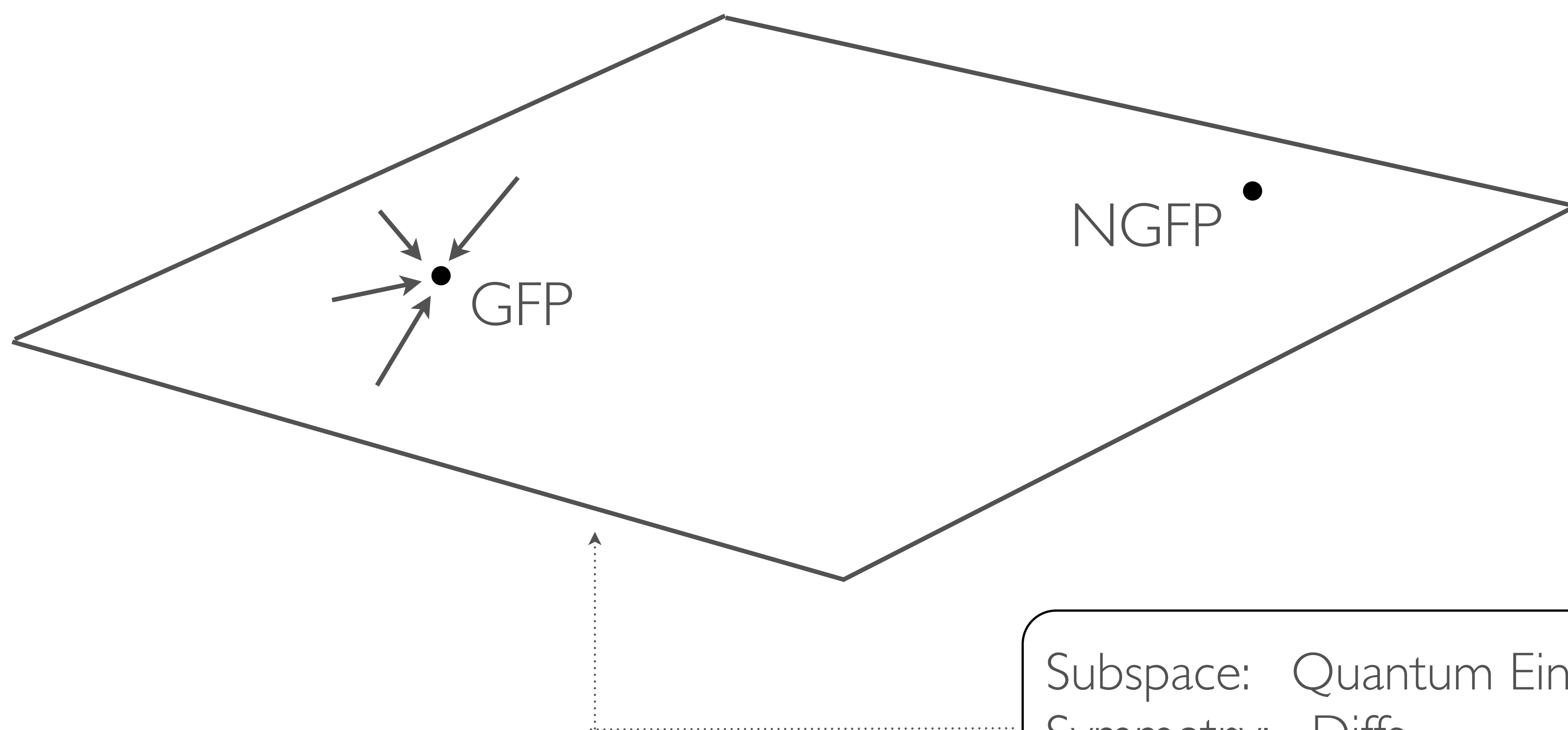
$$N(t,x) = N(t)$$ Projectable HL Gravity

FP Structure of Quantum Gravity

Theory Space: Horava-Lifshitz
Symmetry: Foliation Preserving Diffs





Subspace: Quantum Einstein Gravity
Symmetry: Diffs

Questions

- Is the theory asymptotically free?
- Does it reproduce the correct phenomenology?
- Does ti resolve previous issues?

Questions

- Is the theory asymptotically free?

MATTER-INDUCED FLOWS IN PROJECTABLE HL GRAVITY

Ansatz

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- Projectable Horava-Lifshitz action plus anisotropic scalar

$$\Gamma_k[N, N_i, \sigma, \phi] = \Gamma_k^{\mathrm{HL}}[N, N_i, \sigma] + S^{\mathrm{LS}}[N, N_i, \sigma, \phi]$$

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- Potential V is a function of the intrinsic curvatures:

$$V_k^{(d=2)} = g_0 + g_1 \, R + g_2 \, R^2$$

$$V_k^{(d=3)} = g_0 + g_1 R + g_2 R^2 + g_3 R_{ij} R^{ij} - g_4 R \Delta_x R$$
$$- g_5 R_{ij} \Delta_x R^{ij} + g_6 R^3 + g_7 R R_{ij} R^{ij} + g_8 R^i_j R^j_k R^k_i$$

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$$S^{\mathrm{LS}} \equiv \frac{1}{2}\int dt d^d x N \sqrt{\sigma} \phi \left[\Delta_t + (\Delta_x)^z\right] \phi$$

- Minimally coupled anisotropic scalars with covariant derivatives

$$\Delta_t \equiv -\frac{1}{N\sqrt{\sigma}} \partial_t N^{-1} \sqrt{\sigma} \qquad \Delta_x \equiv -\sigma^{ij} \nabla_i \nabla_j$$

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- We will consider the beta functions for large-n (with n the number of scalars)

ANISOTROPIC HEAT-KERNELS

HK for Anisotropic Operators

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- Need the Heat Kernel for anisotropic operator:

$$\partial_t \Gamma_k = \mathrm{T}r\ \mathcal{O}\left(\mathcal{D}^2\right) \qquad\qquad \mathcal{D}^2 \equiv \Delta_t + (\Delta_x)^z$$

- It is possible to reduce the problem to an Off-Diagonal Heat Kernel computation.

[Benedetti, Groh, Machado, Saueressig JHEP 06 (2011) 079]
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- For a scalar we find:

$$\mathrm{Tr}\, e^{-s\mathcal{D}^2} \simeq (4\pi)^{-(d+1)/2}\, s^{-(1+d/z)/2} \int dt d^d x N \sqrt{\sigma} \times$$

$$\left[\frac{s}{6}\left(e_1\, K^2 + e_2\, K_{ij} K^{ij}\right) + \sum_{n \geq 0} s^{n/z}\, b_n\, a_{2n}\right]$$

$$N = 1, \quad N^i = 0$$

Anisotropic HK: Results

- The e coefficients are:

$$e_1 = \frac{d-z+3}{d+2}\frac{\Gamma(\frac{d}{2z})}{z\Gamma(\frac{d}{2})}\,, \qquad e_2 = -\frac{d+2z}{d+2}\frac{\Gamma(\frac{d}{2z})}{z\Gamma(\frac{d}{2})}$$

- The b coefficients come in two classes:

$$0 \le n \le \lfloor d/2 \rfloor$$

$$b_n = \frac{\Gamma\left(\frac{d-2n}{2z}+1\right)}{\Gamma\left(\frac{d-2n}{2}+1\right)}$$

$$n > \lfloor d/2 \rfloor$$

$$b_n(d,z) \equiv \frac{(-1)^k}{\Gamma(d/2-n+k)} \int_0^\infty dx x^{d/2-n+k-1} \left(\partial_x\right)^k e^{-x^z}$$

$$k = n+1-\lfloor d/2 \rfloor$$

	$d=2$			$d=3$			
	$z=1$	$z=2$	$z=3$	$z=1$	$z=2$	$z=3$	$z=4$
b_0	1	$\frac{\sqrt{\pi}}{2}$	$\Gamma(\frac{4}{3})$	1	$\frac{4}{3\sqrt{\pi}}\Gamma(\frac{7}{4})$	$\frac{2}{3}$	$\frac{4}{3\sqrt{\pi}}\Gamma(\frac{11}{8})$
b_1	1	1	1	1	$\frac{2}{\sqrt{\pi}}\Gamma(\frac{5}{4})$	$\frac{2}{\sqrt{\pi}}\Gamma(\frac{7}{6})$	$\frac{2}{\sqrt{\pi}}\Gamma(\frac{9}{8})$
b_2	1	0	0	1	$\frac{1}{\sqrt{\pi}}\Gamma(\frac{3}{4})$	$\frac{1}{\sqrt{\pi}}\Gamma(\frac{5}{6})$	$\frac{1}{\sqrt{\pi}}\Gamma(\frac{7}{8})$
b_3	1	-2	0	1	$-\frac{2}{\sqrt{\pi}}\Gamma(\frac{5}{4})$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{\pi}}\Gamma(\frac{5}{8})$
b_4	1	0	6	1	$-\frac{4}{\sqrt{\pi}}\Gamma(\frac{7}{4})$	$\frac{9}{2\sqrt{\pi}}\Gamma(\frac{7}{6})$	$\frac{2}{\sqrt{\pi}}\Gamma(\frac{11}{8})$

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- z=1 reproduces standard covariant results

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- z=1 reproduces standard covariant results
- z=d=2 reproduces Baggio, de Boer, Holsheimer, arXiv:1112.6416

RESULTS

Beta Functions

- Defining

$$g_k \equiv G_k\, k^{2\eta}\,, \qquad \eta \equiv \tfrac{d}{2z} - \tfrac{1}{2} \qquad \phi_n \equiv \tfrac{1}{\Gamma(n)} \int_0^1 dx\, x^{n-1}$$

- The beta functions for the dimensionless couplings read

wave-function renormalization

$$\begin{cases} \beta_g = 2\,\eta\, g - \tfrac{2}{3}\,(4\pi)^{-(d-1)/2}\;\phi_\eta\, e_2\, g^2\,, \\ \beta_\lambda = -\,\tfrac{2}{3}\,(4\pi)^{-(d-1)/2}\;\phi_\eta\;(e_1 + \lambda\, e_2)\; g \end{cases}$$

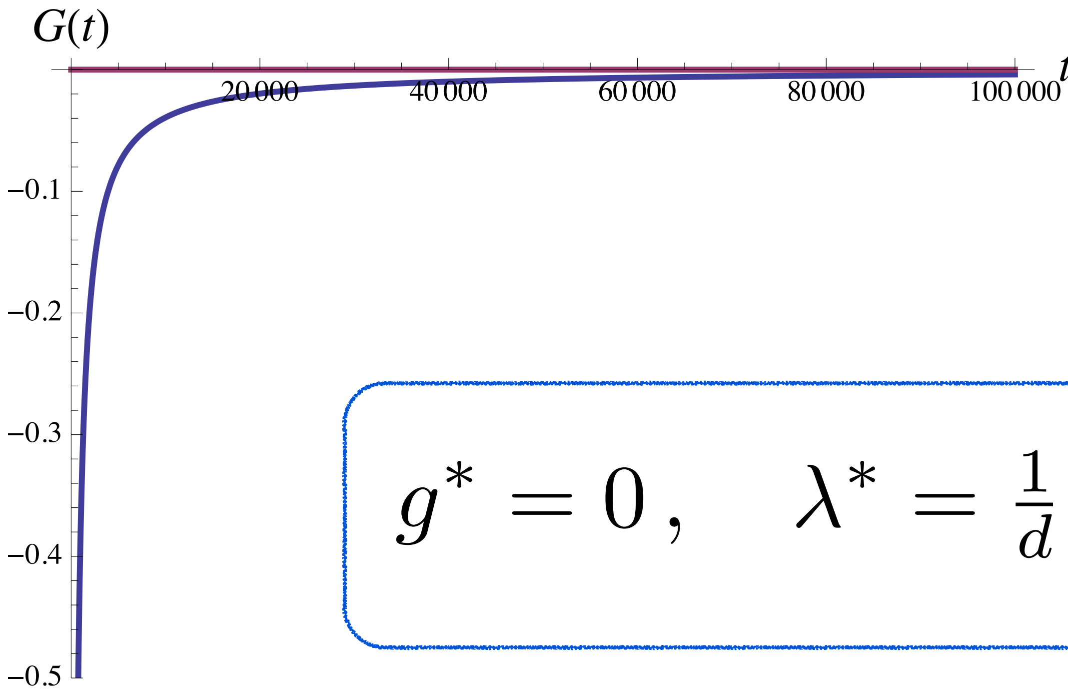
Newton and cosmological constants

$$\begin{cases} \beta_{\tilde g_0} = -\,2\,\tilde g_0 + \frac{4g}{(4\pi)^{(d-1)/2}}\;\left(b_0\,\phi_{\eta+1} - \tfrac{1}{6}\,e_2\,\phi_\eta\,\tilde g_0\right)\,, \\ \beta_{\tilde g_1} = \left(\tfrac{2}{z} - 2\right)\tilde g_1 + \frac{2g}{3(4\pi)^{(d-1)/2}}\;\left(b_1\,\phi_{\eta+1-1/z} - e_2\,\phi_\eta\,\tilde g_1\right) \end{cases}$$

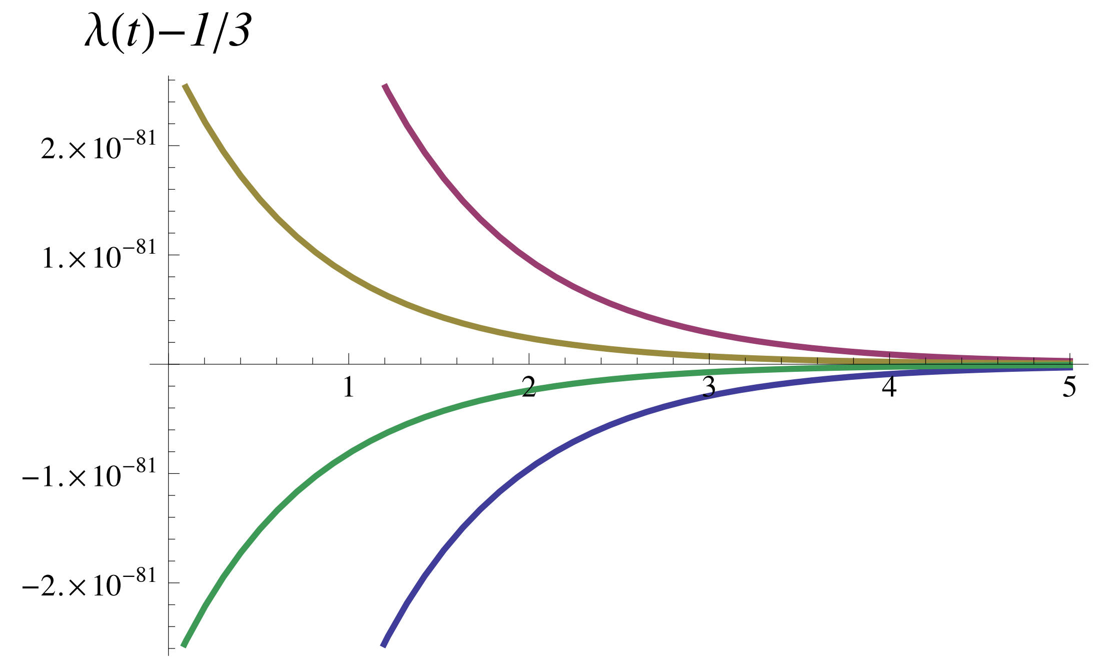
Higher derivative couplings

$$\begin{cases} \beta_{\tilde g_2} = -\,\tfrac{2}{3}\tilde g_2 + \tfrac{g}{5\pi}\left(\frac{1}{8\sqrt{\pi}}\frac{\Gamma(5/6)}{\Gamma(1/3)} + \tilde g_2\right)\,, \\ \beta_{\tilde g_3} = -\,\tfrac{2}{3}\tilde g_3 + \tfrac{g}{5\pi}\left(\frac{1}{4\sqrt{\pi}}\frac{\Gamma(5/6)}{\Gamma(1/3)} + \tilde g_3\right)\,, \\ \beta_{\tilde g_i} = \tfrac{g}{\pi}\left(\tfrac{1}{5}\,\tilde g_i - \tfrac{1}{2}\,c_i\right)\,, \quad i = \{4,5,6,7,8\} \end{cases}$$

Anisotropic Gaussian FP

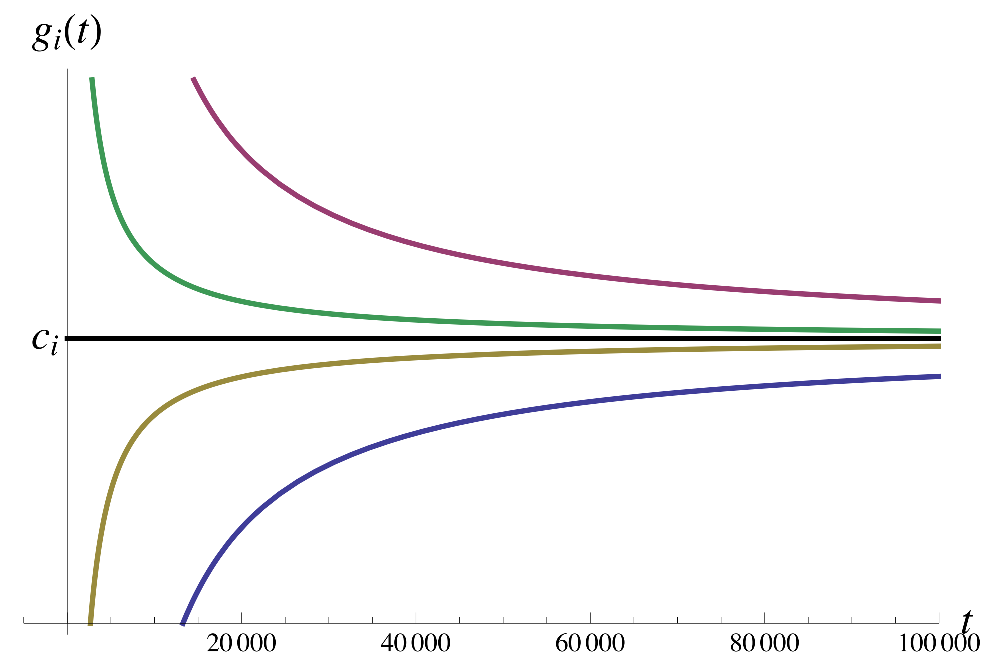


$$g^* = 0\,, \qquad \lambda^* = \frac{1}{d}$$



$$\tilde{g}_1^* = \tilde{g}_2^* = \tilde{g}_3^* = 0\,, \quad \tilde{g}_i^* = \frac{5}{2} c_i$$

$$c_4 = \frac{1}{336}\,, c_5 = \frac{1}{840}\,, c_6 = -\frac{1}{560}\,,$$
$$c_7 = \frac{1}{105}\,, \; c_8 = -\frac{1}{180}$$



Detailed Balance Revisited

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- The Detailed Balance condition states that

$$V \propto \frac{\delta W\left[\sigma\right]}{\delta \sigma_{ij}} H_{ijkl} \frac{\delta W\left[\sigma\right]}{\delta \sigma_{kl}}$$

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- In d=3 this implies that the potential is the square of the Cotton tensor

$$V_{\text{db}} \propto C_{ij} C^{ij} \qquad\qquad C^{ij} = \epsilon^{ikl} D_k \left(R_l^j - \frac{1}{4} R \delta_l^j \right)$$

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$$V_{\rm db} \propto C_{ij} C^{ij} \qquad\qquad C^{ij} = \epsilon^{ikl} D_k \left(R^j_l - \tfrac{1}{4} R \delta^j_l \right)$$

- Constructing the fixed point potential from the fixed point values of the couplings, one can see that <u>it does not respect Detailed Balance</u>

$$V_{\rm db} \propto R_{ij} \Delta_x R^{ij} - \tfrac{3}{8} R \Delta_x R + 3 R^i_j R^j_k R^k_i - \tfrac{5}{2} R R^{ij} R_{ij} + \tfrac{1}{2} R^3$$

VS

$$V_* = g_4^* R \Delta_x R - g_5^* R_{ij} \Delta_x R^{ij} + g_6^* R^3 + g_7^* R R_{ij} R^{ij} + g_8^* R^i_j R^j_k R^k_i$$

CONCLUSIONS

Achievements

- Evaluation of anisotropic Heat Kernels
- Large-N study of HL fixed point

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Open Questions

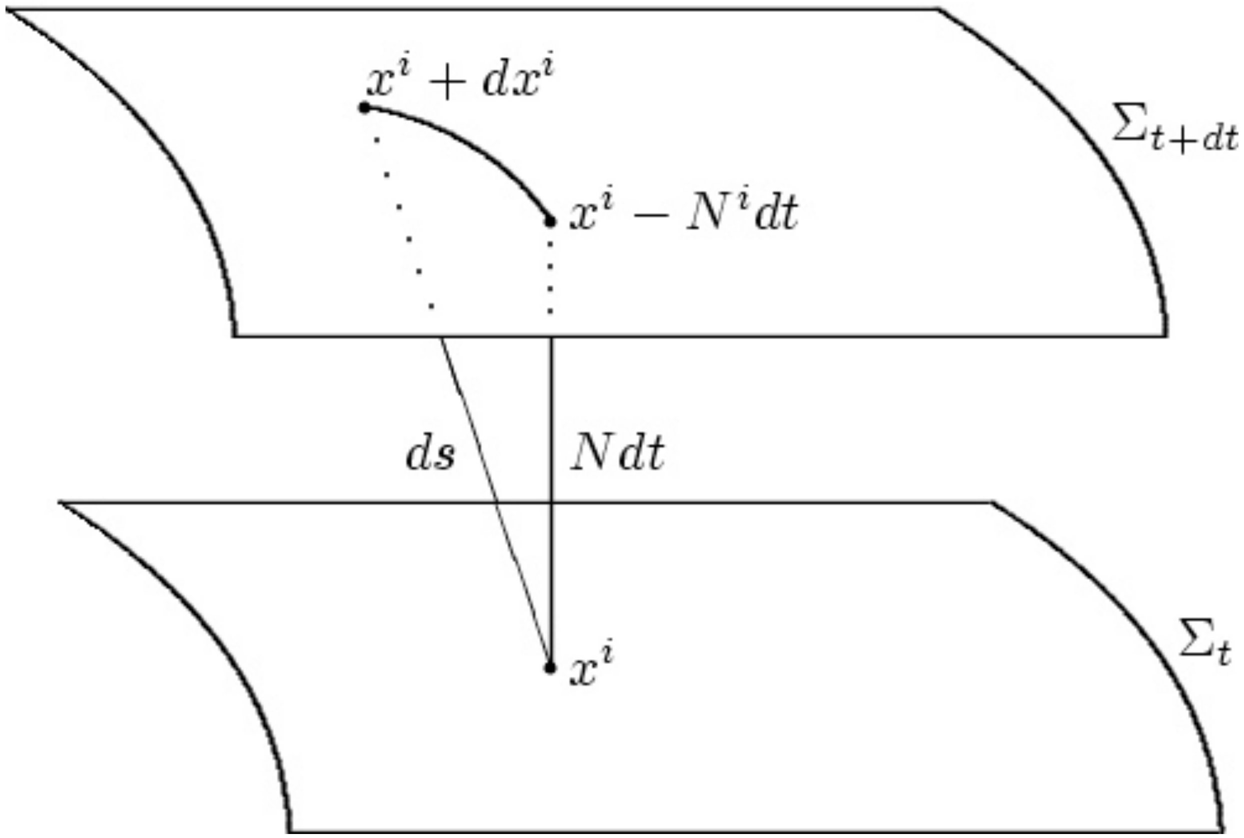
- Flow portrait including higher spins (working on)
- Applications to other systems?

THANK YOU

APPENDIX

More on ADM

- Basic fields: Lapse, Shift, Induced metric $g_{\mu\nu} \mapsto \{\, N \,,\, N_i \,,\, \sigma_{ij} \,\}$
- Induced metric **σ** describes intrinsic curvature on the slice
- Lapse and shift describe extrinsic curvature



$$K_{ij} = \frac{1}{2N}\{-\dot{\sigma}_{ij} + \nabla_i N_j + \nabla_j N_i\}$$

- EH action in ADM form (Gauss-Codazzi relation)

$$S_{EH} = \int \{(K^{ij}K_{ij} - K^2) + R\}\sqrt{\sigma}\; N\; d^3x\; dt$$

Evaluating the Trace

- Notice that on general backgrounds: $[\Delta_t, \Delta_x] \neq 0$
- Laplace transform:

$$\mathrm{Tr} f\left(\mathcal{D}^2\right) \to \mathrm{Tr}\, e^{-s\mathcal{D}^2}$$

- Split the exponential using the inverse Baker-Hausdorf (Zassenhaus) formula:

$$e^{t(X+Y)} \simeq e^{tX}\, e^{tY} \boxed{e^{-\frac{t^2}{2}[X,Y]}\, e^{\frac{t^3}{6}(2[Y,[X,Y]+[X,[X,Y]])} \dots}$$

$$[\Delta_t, (\Delta_x)^z]$$

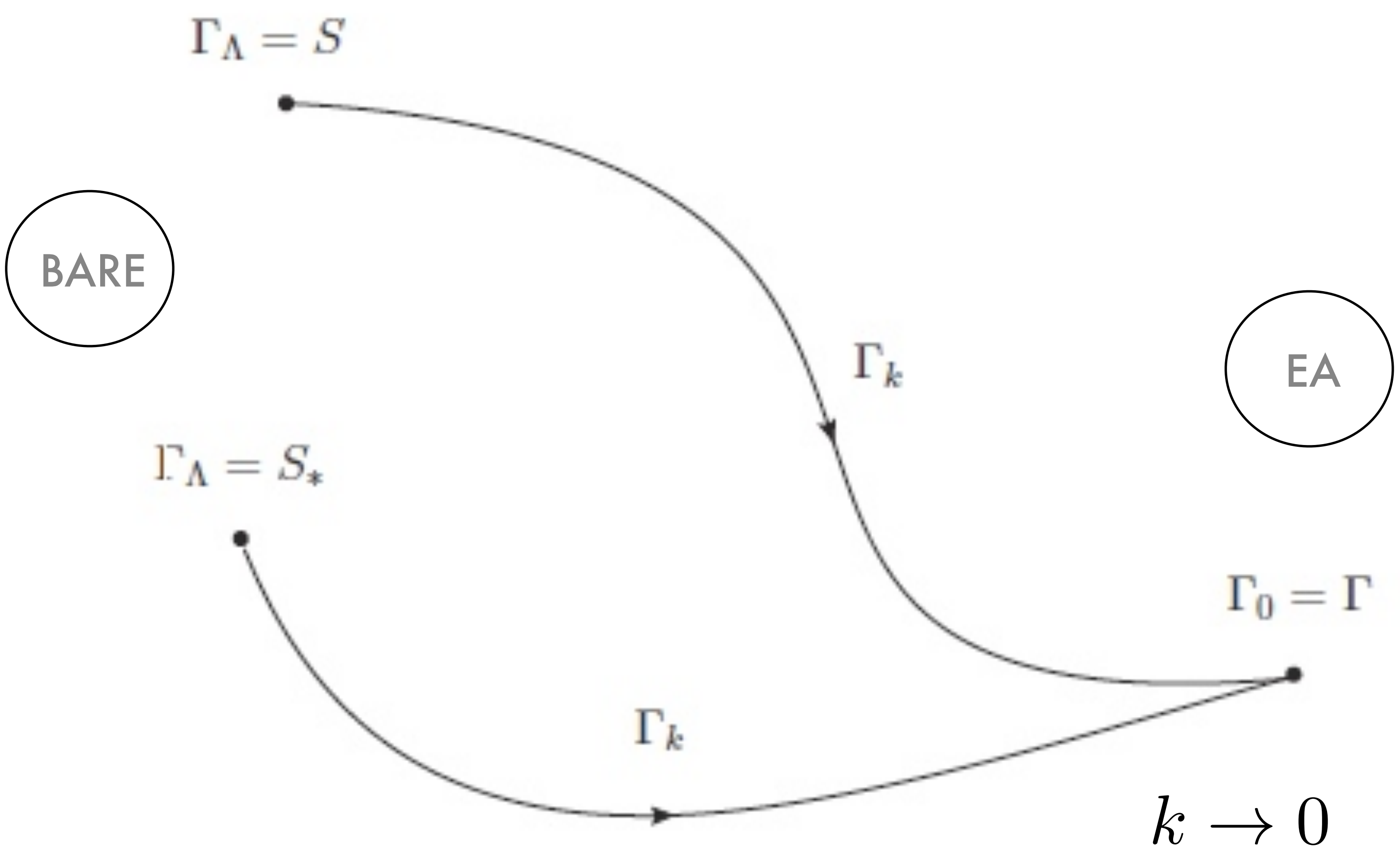
- Do another Laplace transform involving only the spatial Laplacian, and use Baker-Hausdorf again

- End up with: $exp\left(-s_1\Delta_t - s_2\Delta_x\right)$

- After a scale transformation, resum the two Laplacians into a "fake" covariant one. Then use Off-Diagonal techniques

Theory Space

The EAA interpolates smoothly between the bare action and the full quantum EA



$$\lim_{k\to\infty} \Gamma_k[\varphi] = S[\varphi]$$

$$\lim_{k\to 0} \Gamma_k[\varphi] = \Gamma[\varphi]$$

Asymptotic Safety

- Gravity is perturbatively nonrenormalizable $[G_N] = -2$
- UV-completion:
  1. New physics
  2. Nonperturbative "self-healing"

Asymptotically Safe Theory:

- Has a (non-gaussian) RG fixed point
- The UV Critical Surface is finite dimensional

- Generalized, nonperturbative renormalizability requirement