Fixed Functionals in Quantum Einstein Gravity

Maximilian Demmel, Frank Saueressig, Omar Zanusso

THEP
Uni Mainz

22.09.2014

1 introduction

2 pole crossing - analytic example

3 fixed functions in $d = 3$

4 fixed functions in $d = 4$

5 summary

setup

- tool: $k\frac{\mathrm{d}\Gamma_k}{\mathrm{d}k} = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\frac{\mathrm{d}\mathcal{R}_k}{\mathrm{d}k}\right]$
- truncation: $\Gamma_k^{\mathrm{grav}}[g_{\mu\nu}] = \int \mathrm{d}^d x\,\sqrt{g}\; f_k(R)$
  [0705.1769, 0712.0445, 1204.3541, 1211.0955, ...]
- background field method $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$
- conformal reduction: $h_{\mu\nu} = \frac{1}{d}\bar{g}_{\mu\nu}\,\phi$
  [0801.3287, ...]
- maximally symmetric background: S^d $(R > 0)$

type II regulator

operator: $\Box := -\bar{D}^2 + \mathbf{E}$ (potential term $\mathbf{E}$ containing $\bar{R}$)

define regulator $\mathcal{R}_k(\Box)$

$$\Gamma_k^{(2)}(\Box) + \mathcal{R}_k(\Box) \stackrel{\text{def.}}{=} \Gamma_k^{(2)}(\Box + R_k(\Box)),$$

where R_k is the profile function (Litim's cutoff).

flow equation

$$\int \mathrm{d}^d x \sqrt{g}\, \partial_t f_k(\bar{R}) = \frac{1}{2} \,\mathrm{Tr}\, W(\Box)$$

operator trace

spectral sum:

$$\mathrm{Tr}\, W(\Box) = \sum_i D_i W(\lambda_i)$$

eigenvalues λ_i and multiplicities D_i of $\Box$

integrating out eigenmodes:

$$\begin{aligned} \text{optimised cutoff} &\implies W(\lambda_i) \propto \theta(k^2 - \lambda_i) \\ &\implies W(\lambda_i) \neq 0 \iff k^2 \geq \lambda_i \end{aligned}$$

- every time k crosses λ_i the eigenmode is integrated out
- spectral sum is a finite sum (ACHTUNG: $\lambda_i \propto R$)

definition: fixed function

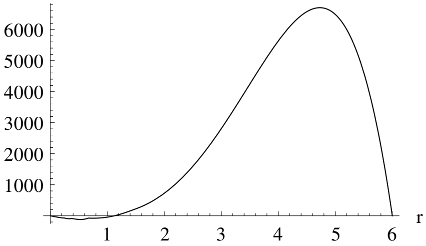
definition:

$$R =: k^2 r, \qquad E =: k^2 e, \qquad f_k(R) =: k^d \varphi_k(R/k^2)$$

- flow equation: partial differential equation for $\varphi_k(r)$
- fixed functions: (global) k stationary solutions $\iff \partial_t \varphi_k(r) = 0$
- third order equation: three parameter family of solutions

pole structure of flow equation

the coefficient of $\varphi'''(r)$



- three poles r_0, r_1 and r_2 (in $d = 3, 4$)
- global solutions should cross the poles

regular singular points and pole crossing

generic ODE: $y^{(n)}(x) = f(y^{(n-1)}, \dots, y', y, x)$

r.h.s. f can have singular points

$$f(y^{(n-1)}, \dots, y', y, x) = \frac{e(y^{(n-1)}(x_0), \dots, y'(x_0), y(x_0), x_0)}{x - x_0} + \mathcal{O}\left((x - x_0)^0\right)$$

- pole crossing (global) solution $\iff$ $e\big|_{x=x_0} = 0$ (regularity condition)
- additional boundary condition reduces number of free parameters

analytic example

initial value problem:

$$y''(x) = -\frac{y(x)}{x-1}, \qquad y(0) = y_0,\ y'(0) = y_1$$

solutions are modified Bessel functions I_n. For most y_0, y_1: no global solution.

add BC y(1)=0: pole crossing solution

$$y(x; y_0) = y_0 \frac{\sqrt{1-x}\, I_1\left(2\sqrt{1-x}\right)}{I_1(2)}$$

self similarity in initial values $y(x; y_0) = \lambda^{-1} y(x; \lambda y_0)$

analytic example

initial value problem:

$$y''(x) = -\frac{y(x)}{x-1}, \qquad y(0) = y_0,\ y'(0) = y_1$$

polynomial expansion: $y(x) = \sum_{n=0}^{k} a_n x^n$

- fixing all coefficients at $x = 0 \implies a_i = 0$
- improved strategy: fix a_n at $x = 0$ for $n \geq 2$ and fix a_1 at $x = 1$ $\implies$ series expansion of analytic solution recovered
- self similarity $y(x) = y_0 \sum_{n=0}^{k} \tilde{a}_n x^n$

numerical shooting I

expand around $r = 0$

$$\varphi(r; a_0, a_1) = a_0 + a_1 r + \sum_{n=2}^{k} a_n(a_0, a_1) r^n$$

1. fix a_n, $n \geq 2$ by using regularity condition at $r = 0$

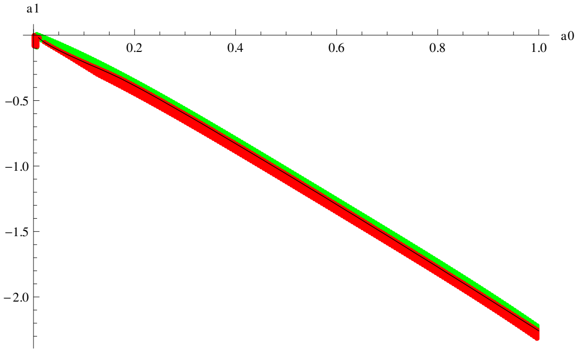
2. initial conditions for numerical integration $(\varepsilon > 0)$

$$\varphi^{(n)}_{\text{init}}(\varepsilon) = \varphi^{(n)}(\varepsilon; a_0, a_1), \qquad n = 0, 1, 2$$

3. regularity condition at r_1

$$e : (a_0, a_1) \mapsto \mathbb{R}$$

pole crossing solutions



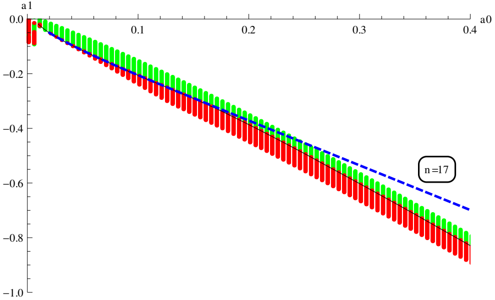
- green: $e(a_0, a_1) > 0$
- red: $e(a_0, a_1) < 0$
- black line: $e(a_0, a_1) \approx 0$

polynomial expansion

- polynomial expansion

$$\varphi(r) = \sum_{k=0}^{n} a_k r^k$$

- boundary condition at $r = 0$ $(a_2, a_3, \dots)$
- boundaty condition at $r = r_1$ (a_1)
- a_1 as function of a_0 (dashed blue line)



numerical shooting II

algorithm for second shooting

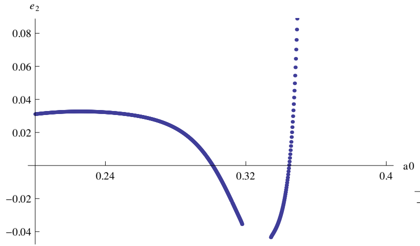
1. parametrize regular line by a_0
2. initial conditions for second numerical integration

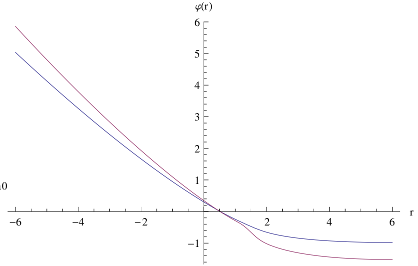
$$\varphi^{(n)}_{\text{init}}(r_1 + \varepsilon) = \varphi^{(n)}_{\text{num}}(r_1 - \varepsilon; a_0), \qquad n = 0, 1, 2$$

3. numerically integrate up to r_2 and check regularity condition

$$e_3 : a_0 \mapsto \mathbb{R}$$

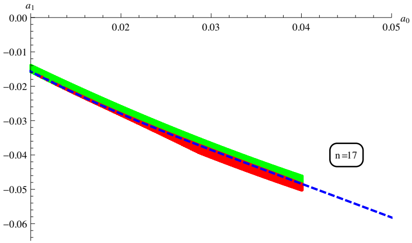
regularity at r_2

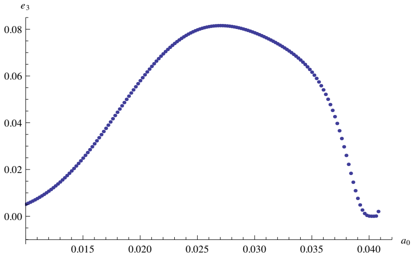




- There are two distinct zeros $\Longrightarrow$ two fixed functions
- improved stability: at most three relevant directions

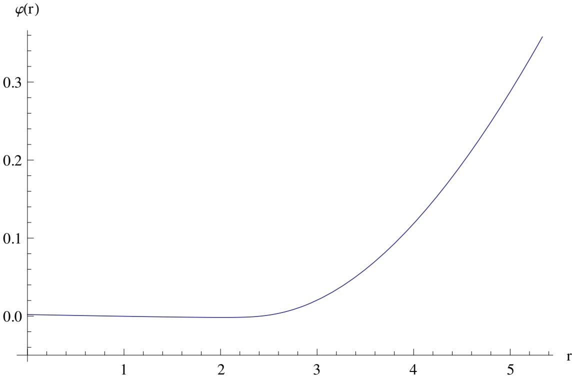
preliminary results in $d = 4$





- numerical shooting and polynomial expansion yield regular line
- one (nontrivial) fixed function identified

fixed function in $d = 4$



summary

- flow of conformally reduced $f(R)$-gravity
- interpretation of integrating out eigenmodes
- numerical and analytical techniques for pole crossing
- two fixed functions in $d = 3$
- one fixed function in $d = 4$

Porto Katsiki - 21. Sept. 2014

Thank You!