Derivative expansion: numerical results at order six, convergence and optimization

with L. Canet, H. Chaté and I. Balog
(to be published.....)

Questions: - Does the DE converge?

- Can we optimize it?

Many papers on this subject (D. Litim, J. Pawlowski
L. Canet, H. Chaté, D. Mouhanna
M. Tissier, J. Vidal)

but ... not a well-posed question!

⇝ e.g.: perturbation theory: not convergent!

⇒ at best asymptotic and Borel-summable
(but QED works well!)

⇒ in this case, convergence means:

convergence of the resummed Padé - Borel (for instance) series!

⇝ optimization w.r.t. which criterion?

Optimization before convergence

two kinds of optimization $\begin{cases} \text{"global"} \\ \text{"local"} \end{cases}$

<u>global</u>: optimize generic properties of the RG flow through the choice of cut-off function $R_k(q)$, independently of the physical quantity studied.

<u>local</u>: choose $R_k(q)$ to optimize a given physical quantity at a given order of the DE.

<u>Example</u>: φ^4, $N=1$, $d=3$, LPA

* Litim's claim on global optimization:

There exists regulators that are "optimal"

e.g.:

$$R_k(q) = \frac{3.92\, q^2}{e^{q^2/k^2} - 1} \quad ; \quad R_k(q) = (k^2 - q^2)\,\Theta(k^2 - q^2)$$

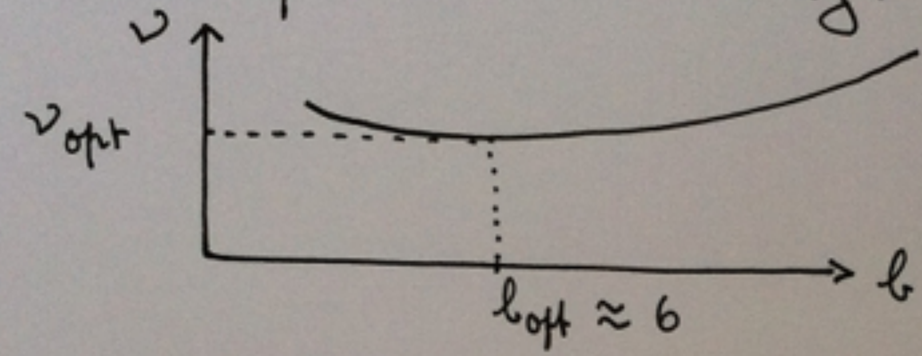
$$R_k(q) = k^4/q^2 \quad ; \quad \ldots\ldots$$

* "local" optimization of the critical exponent ν:

① choose a family of cut-off functions

e.g. $$R_k(q) = b\,\frac{q^2}{e^{q^2/k^2}-1}$$

② Compute (numerically) $\nu(b)$



Principle of Minimal Sensitivity (PMS)

③ Repeat "ad nauseam" steps ① and ② with different families of R_k and compare the different ν_{opt} (for instance $R_k(q) = b(k^2-q^2)\,\theta(k^2-q^2) \Rightarrow b_{opt}=1$)

Results

* at LPA, ν_{opt} is always a minimum and above ν_{exact}.

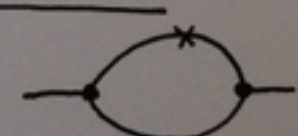
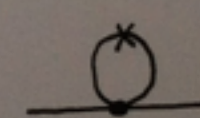
* $$\begin{cases} \nu_{opt}^{exp.} = 0.6503 \\ \nu_{opt}^{\theta} = 0.6494 \end{cases}$$ very close!

(Canet 2004)

<u>Questions:</u>

- Does the PMS apply at each order of the DE and for any physical quantity? (one and only one extremum)
- Does the extremum point to the right direction?
- Does the series of numbers ν^{opt}_{LPA}, $\nu^{opt}_{O(2)}$, $\nu^{opt}_{O(4)}$, $\nu^{opt}_{O(6)}$ seem to converge to the exact result?
- Does the series of numbers ν^{exp}_{LPA}, $\nu^{exp}_{O(2)}$, $\dots$, $\nu^{exp}_{O(6)}$ and idem $\nu^{\theta}_{LPA}, \dots, \nu^{\theta}_{O(6)}$ converge to the same number?

<u>A global criterion</u>

$\partial_t \Gamma_k^{(2)} =$  $+$ 

in general a graph $= \int_q \partial_k R_k(q) \, (\text{Propag})^n (\text{vertex})^p$

$$DE|_m \Rightarrow \begin{cases} \text{vertex} = \text{polynomial in momenta of order } m \\ \text{propag} = \dfrac{1}{\Gamma_k^{(2)}(q) + R_k(q)} = \dfrac{1}{\text{Polyn}(q) + R_k(q)} \end{cases}$$

$\rightsquigarrow$ finite radius of convergence of these expansions in p_i/k

$\Rightarrow$ very bad beyond this radius of conv.

$\Rightarrow$ need to cut-off <u>efficiently</u> for $p \gtrsim k$.

Best choice : $R_k(q>k) \equiv 0$?

e.g. $R_k(q) = (k^2 - q^2)\,\theta(k^2 - q^2)$?

No because this cut-off does not regularize the DE at "large" orders.

$$\rightsquigarrow \quad R_k(q) \propto k^2 \left(1 - \frac{q^2}{k^2}\right)^n \theta(k^2 - q^2)$$

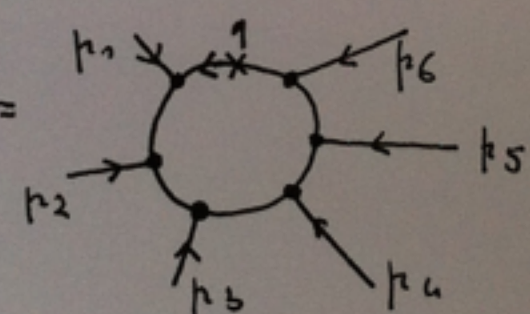
<u>Remark</u> : $\forall$ order of DE , $\exists\, n$ / DE is regularized

BUT...

$\forall\, n$, $\exists$ order of DE / DE is not regul.

<u>Remark</u>: at order $O(\partial^6)$ of DE, we need:

$\partial_t \Gamma_k^{(6)} =$



$+ \dots$

$\Rightarrow$ a product of 6 $\Gamma_k^{(3)}$ functions

$\Rightarrow$ $p_1 p_2 \cdots p_6\, q^{30}$

$\Rightarrow$ difficult to make accurate and controlled calculations with q^{30}!

$\Rightarrow$ keep only terms up to $(\text{momentum})^6$.

Ansatz at order $O(\partial^6)$, regulator, η_k

① $$\Gamma_k[\phi] = \int d^dx \Big\{ U_k + \frac{1}{2} Z_k (\partial_\mu \phi)^2$$
$$+ \frac{1}{2} W_a (\partial_\mu \partial_\nu \phi)^2 + \frac{1}{2} W_b \, \phi \, \partial^2 \phi (\partial_\mu \phi)^2$$
$$+ \frac{1}{2} W_c (\partial_\mu \phi)^4$$
$$+ \frac{1}{2} X_a (\partial_\mu \partial_\nu \partial_\rho \phi)^2 + \cdots + \frac{1}{96} X_h \left((\partial_\mu \phi)^2 \right)^3 \Big\}$$

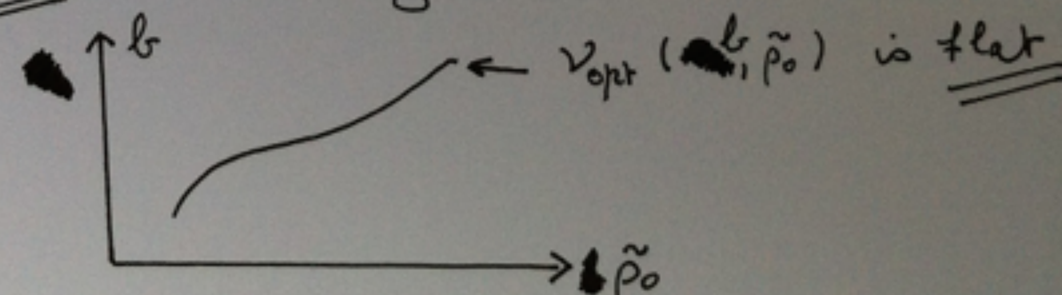
② $R_k(q) = b \, Z_k \dfrac{q^2}{e^{q^2/k^2} - 1}$, $R_k(q) = b \, Z_k k^2 \left(1 - \dfrac{q^2}{k^2}\right)^n \theta(k^2 - q^2)$

(we have also studied three other families of R_k functions).

③ compute the running $\eta_k = -k \partial_k \log Z_k$ by imposing a renormalization prescription on $Z_k(\rho = \frac{\phi^2}{2})$ at a given ρ_0 : $Z_k(\rho) = Z_k \, \mathfrak{z}_k(\tilde\rho) \to \mathfrak{z}(\tilde\rho_0) = 1$.

$\Rightarrow$ <u>two</u> kinds of arbitrariness $\begin{cases} \text{- choice of } R_k \\ \text{- choice of } \rho_0 \end{cases}$

<u>No</u> $\rightsquigarrow$ only <u>one</u>



One can prove that changing b or $\tilde\rho_0$ is equivalent (equivalence theorem? reparametrization invariance?)

Ising : exact results in $d=3$

El-Schwok, Rychkov
Paulos, Poland, Simmons,
Vichi 2014

$$\nu = 0.62999(5)$$
$$\eta = 0.03631(3)$$
$$\omega = 0.8303(18)$$

our results

* exponential cut-off:

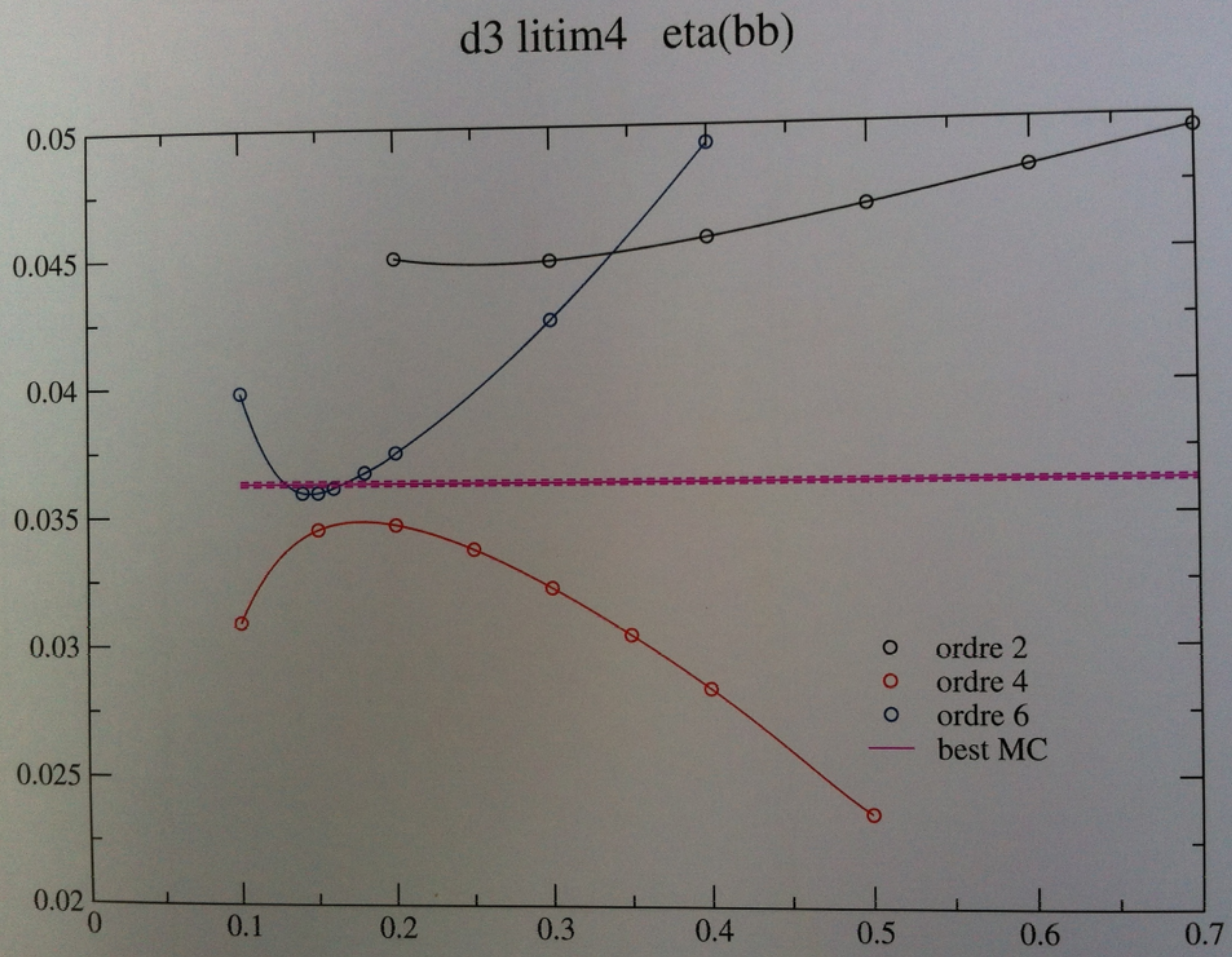
	LPA	$O(\partial^2)$	$O(\partial^4)$	$O(\partial^6)$
ν	0.6503	0.6281	0.63031	0.6304
η	0	0.04427	0.03454	0.03580
ω		0.872		

* Θ - cut-off

ν		$m=3$	0.6302	
		$m=4$	0.6302	0.6303
		$m=5$	0.6303	0.6303
		$m=6$	0.6303	0.6303
η		$m=3$	0.03507	
		$m=4$	0.03481	0.03589
		$m=5$	0.03458	0.03589
		$m=6$	0.03445	0.03601
ω	≈ 0.87			$m=6$ 0.826

Other results

6 - loop : $\nu = 0.6304(13)$, $\eta = 0.0335(25)$, $\omega = 0.799(11)$
HT : $\nu = 0.63012(16)$, $\eta = 0.03639(15)$, $\omega = 0.825(50)$
MC : $\nu = 0.63002(10)$, $\eta = 0.03680(20)$, $\omega = 0.821(5)$


d3 litim4 eta(bb)
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
ordre 2
ordre 4
ordre 6
best MC