CP3 Origins

ERG and Weyl invariance

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[1210.3284, 1312.7097, ...]

ERG2014 Lefkada

Outline of the talk

RG theory: what we know about the flow?

Fixed points and Wess-Zumino actions

Away from criticality

Exact RG flow for the c- and a-functions

Weyl consistency conditions and Local RG

Some examples of approximated c- and a-functions

Switch on gravity!

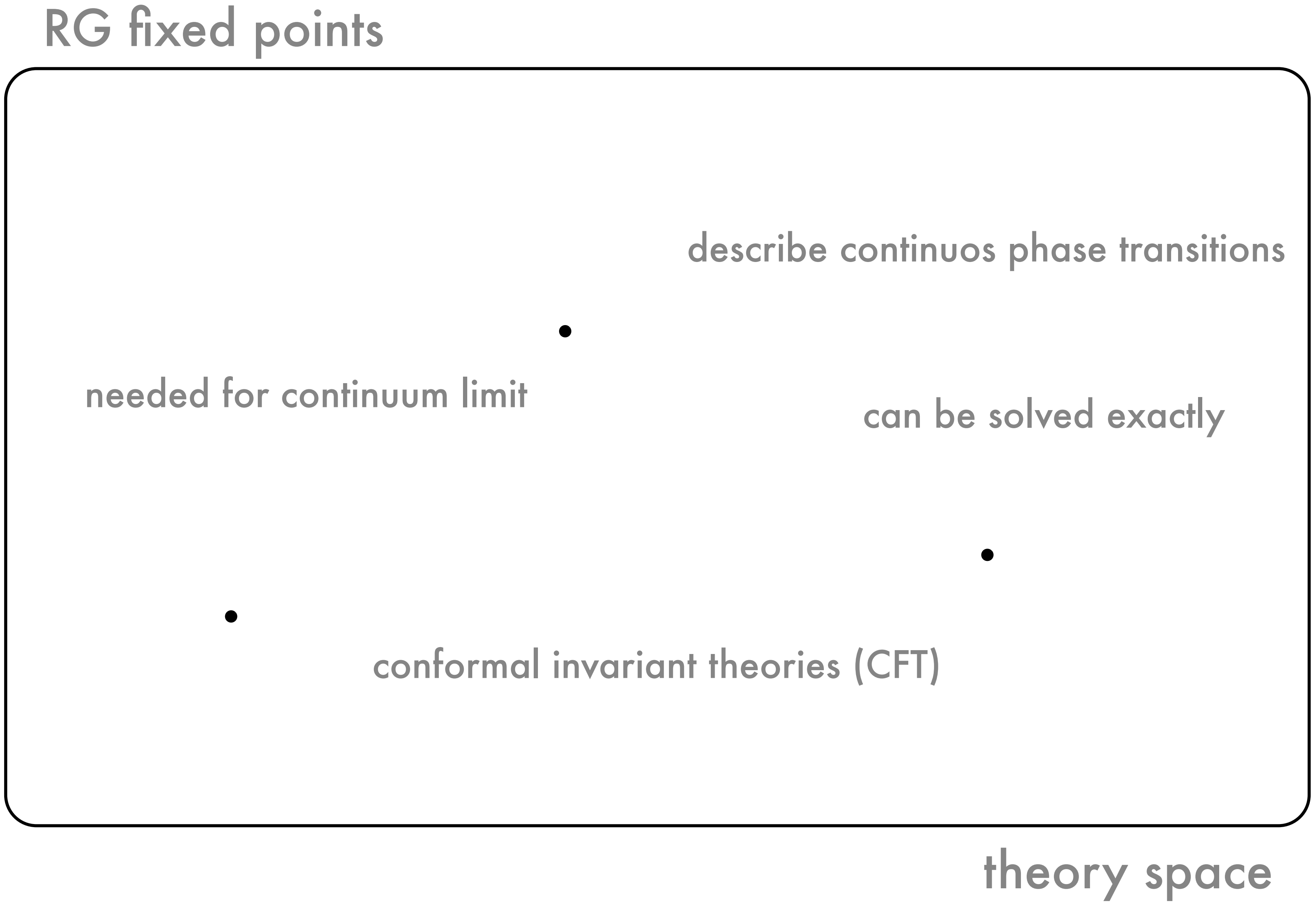
RG theory

RG flow

every theory consistent with the symmetries

theory space

RG theory



RG fixed points

describe continuos phase transitions

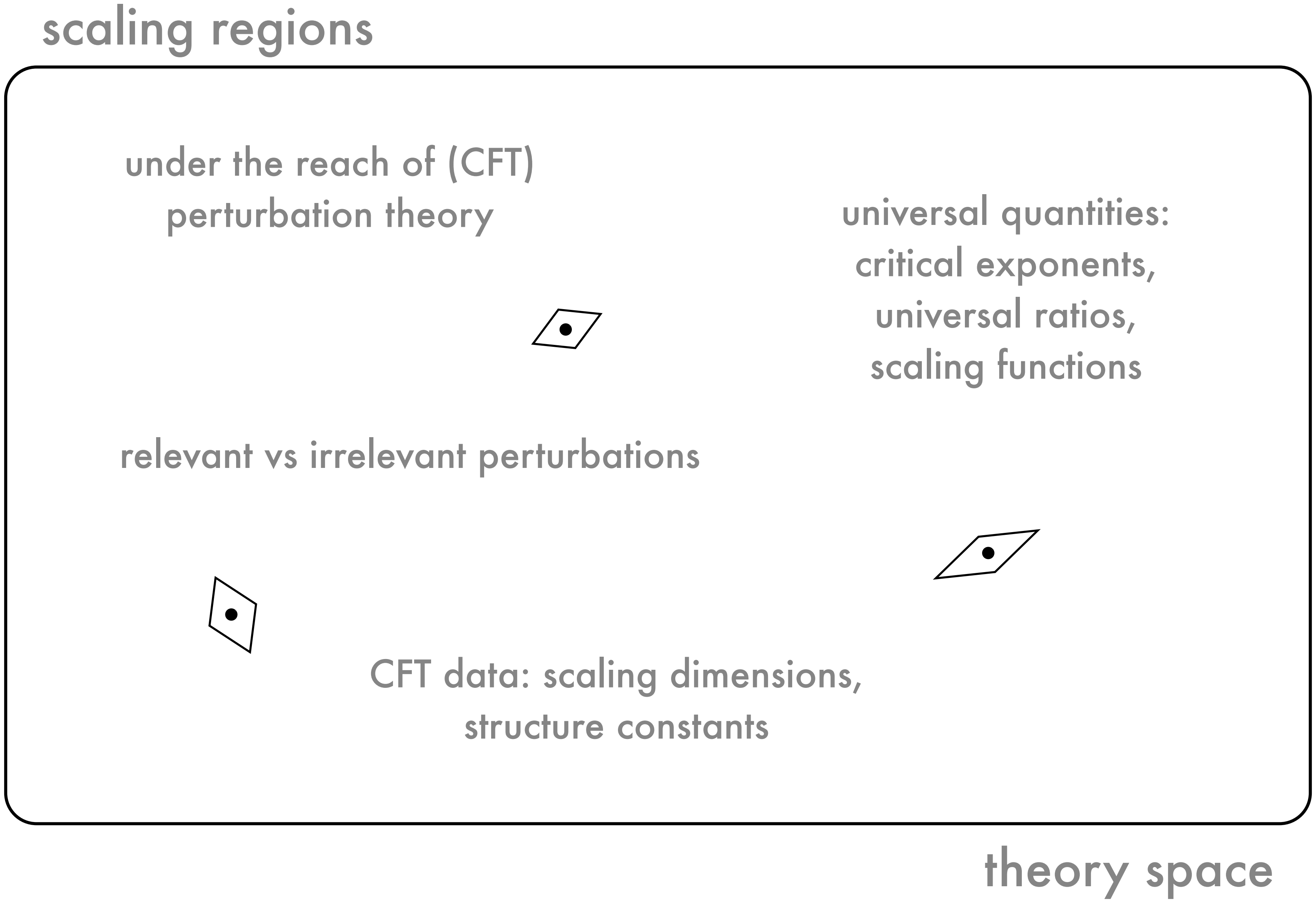
needed for continuum limit

can be solved exactly

conformal invariant theories (CFT)

theory space

RG theory



scaling regions

under the reach of (CFT)
perturbation theory

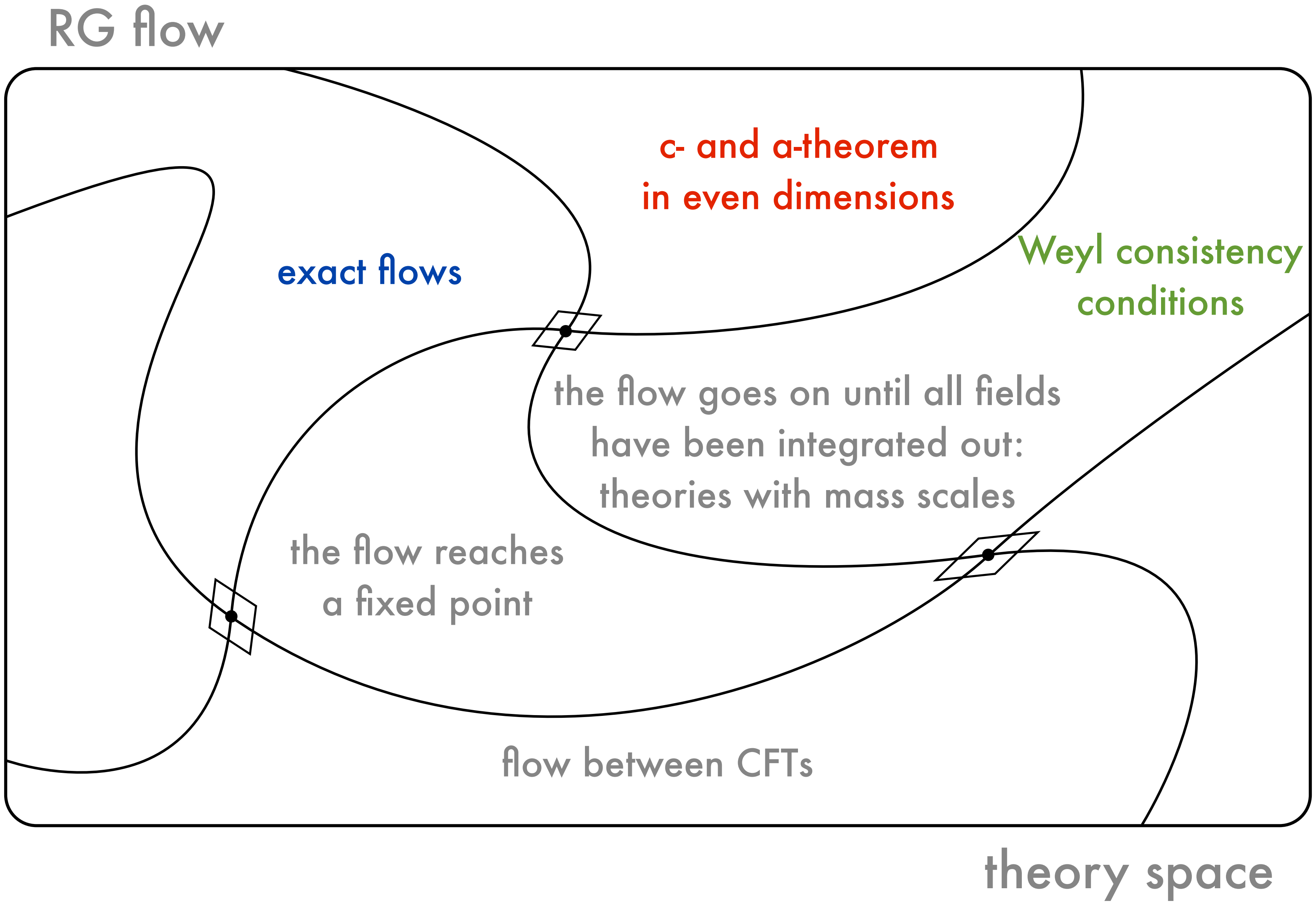
universal quantities:
critical exponents,
universal ratios,
scaling functions

relevant vs irrelevant perturbations

CFT data: scaling dimensions,
structure constants

theory space

RG theory



RG flow

c- and a-theorem
in even dimensions

exact flows

Weyl consistency
conditions

the flow goes on until all fields
have been integrated out:
theories with mass scales

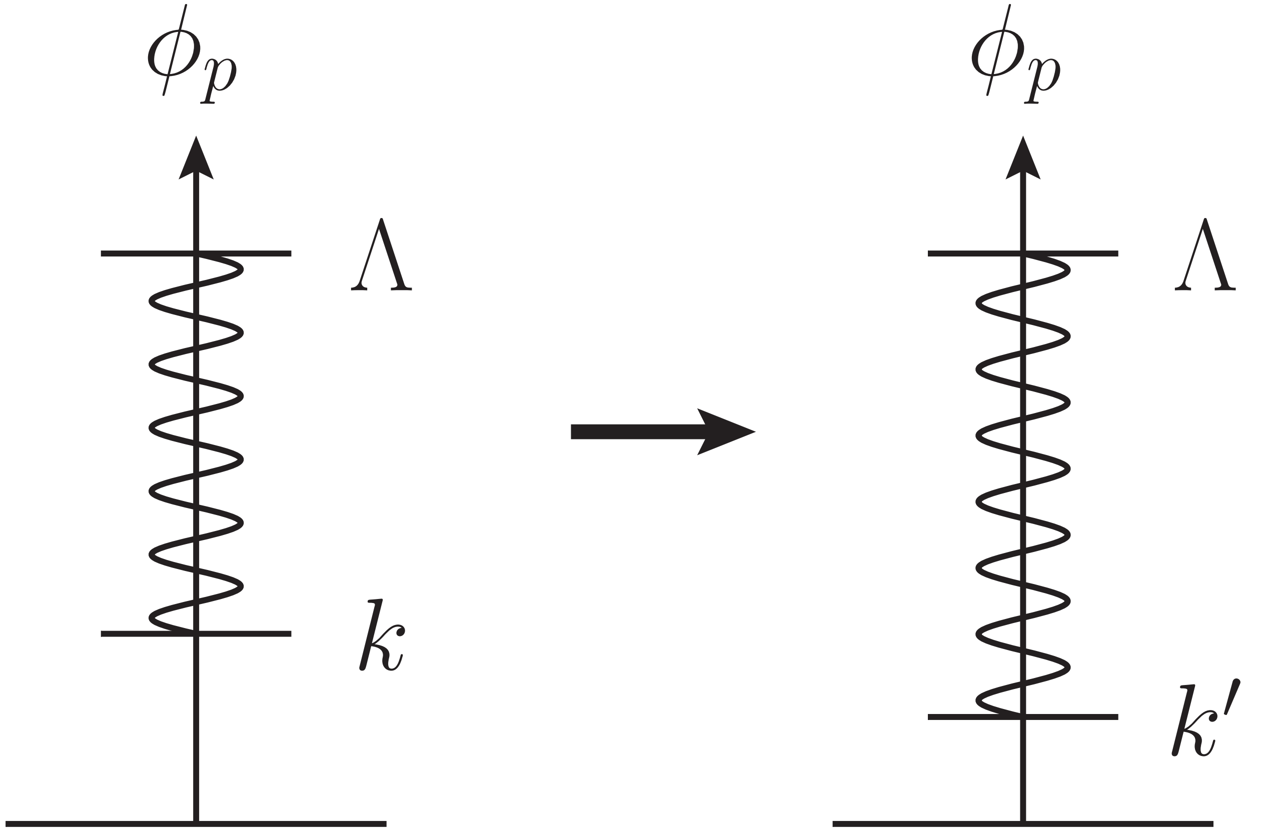
the flow reaches
a fixed point

flow between CFTs

theory space

Exact RG flows

the path integral is a sum over field modes: do it step by step!

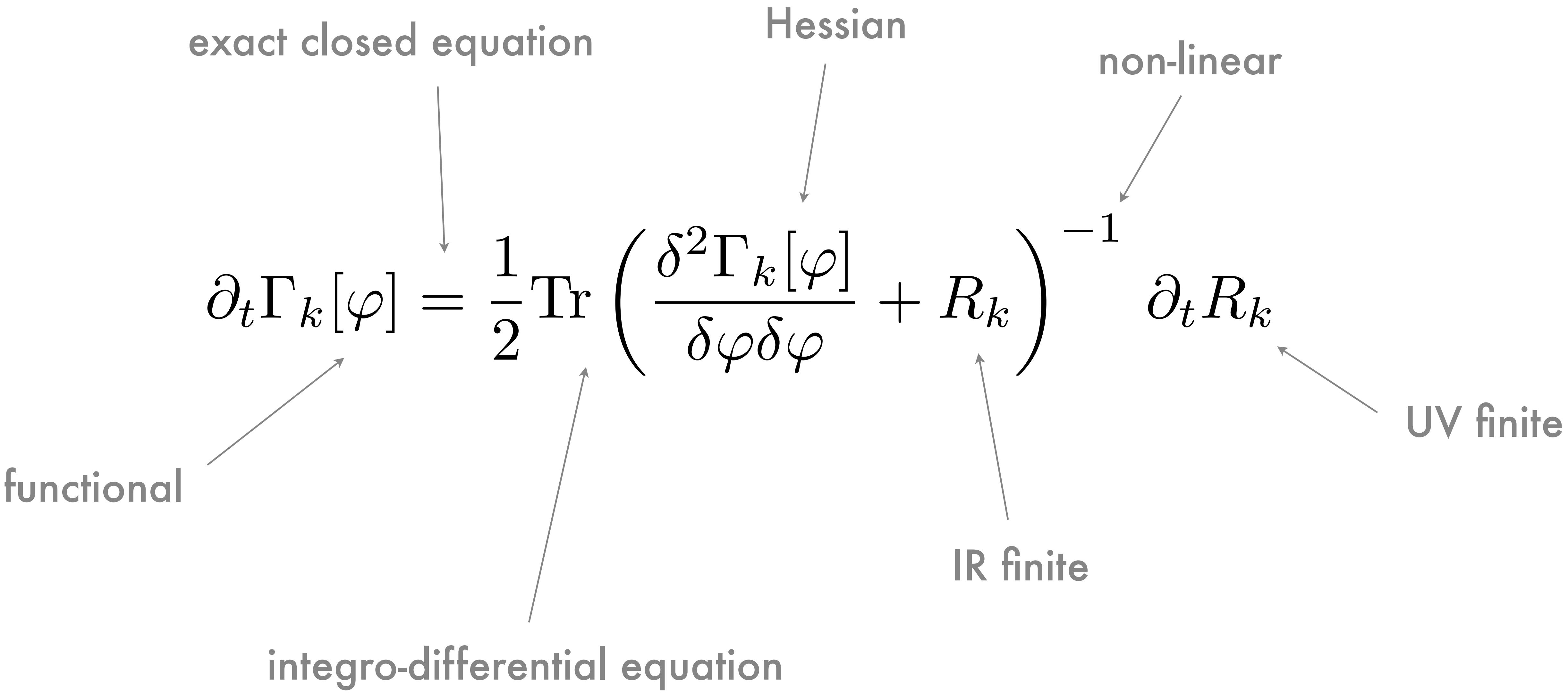


$$\Gamma_k \quad \to \quad \Gamma_{k'} \quad \to \quad \Gamma_{k''} \quad \to \quad \cdots$$

the RG flow is generated by varying the IR scale

Exact RG flows

Anatomy of an equation:



exact closed equation

Hessian

non-linear

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k[\varphi]}{\delta\varphi\delta\varphi} + R_k \right)^{-1} \partial_t R_k$$

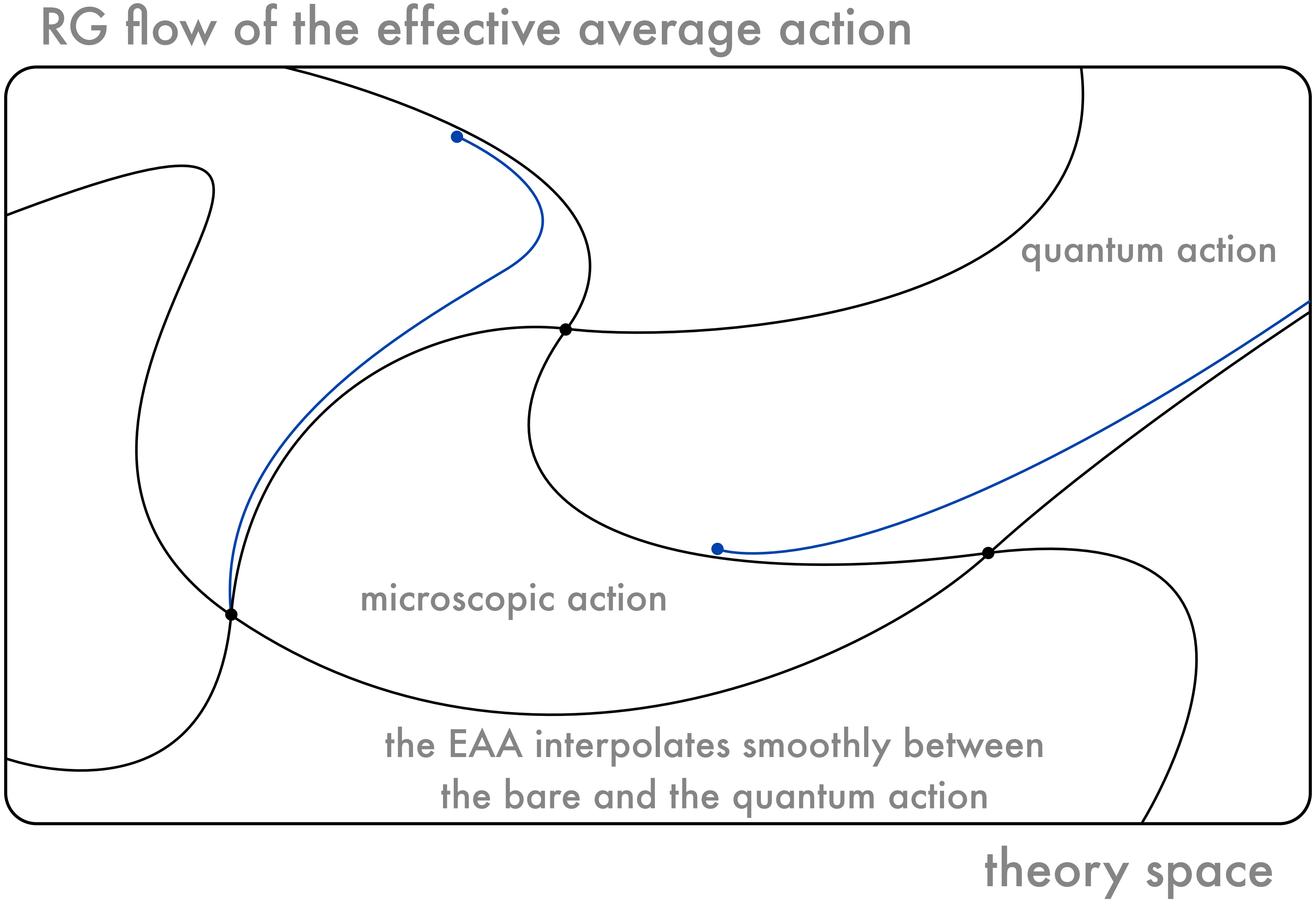
UV finite

functional

IR finite

integro-differential equation

Exact RG flows


RG flow of the effective average action

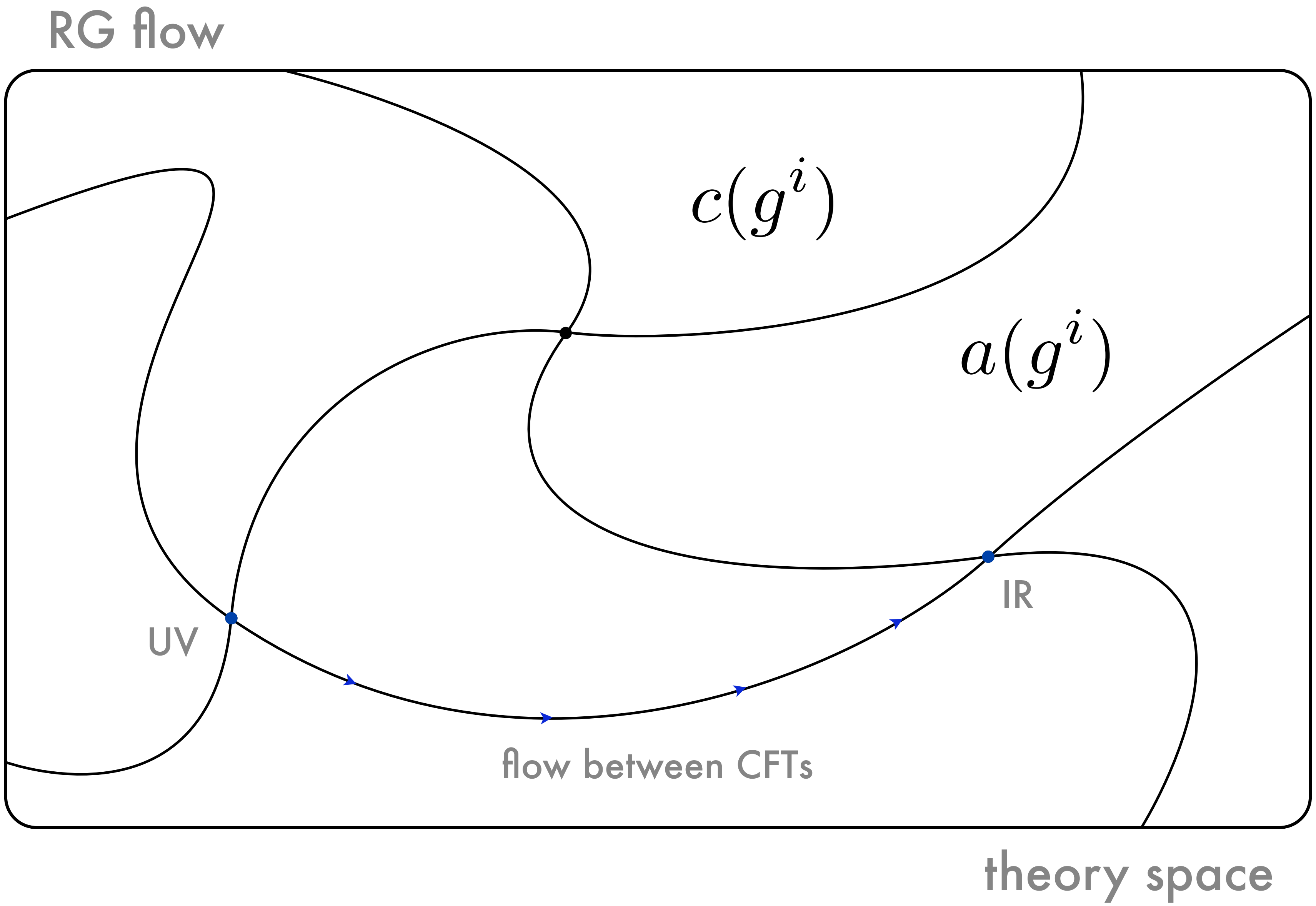
quantum action

microscopic action

the EAA interpolates smoothly between
the bare and the quantum action

theory space

c- and a-theorem

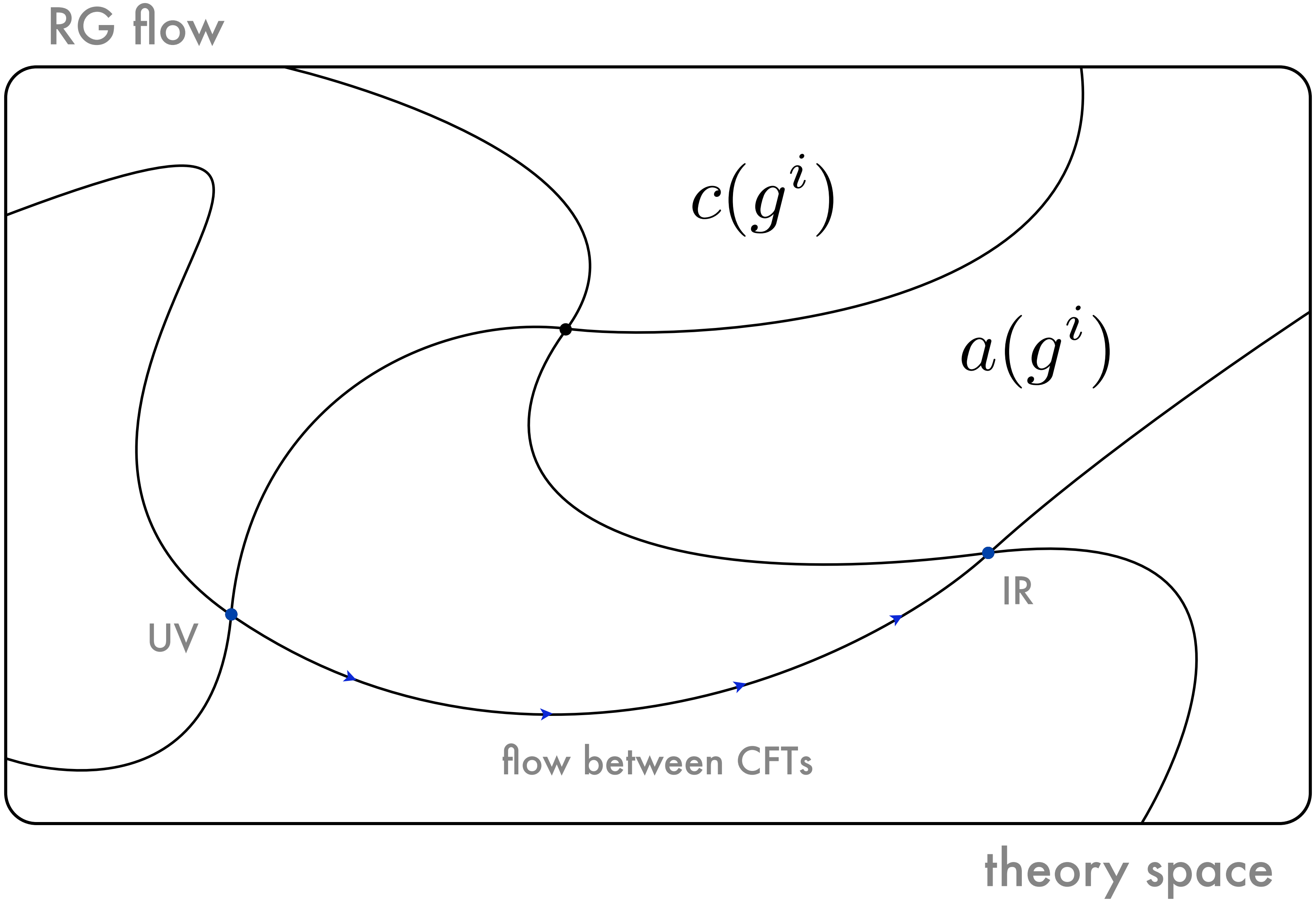


Δc and Δa are universal quantities depending on a whole trajectory!

Integrated (or weak) c- and a-theorems:

$$\Delta c > 0 \qquad \Delta a > 0$$

c- and a-theorem

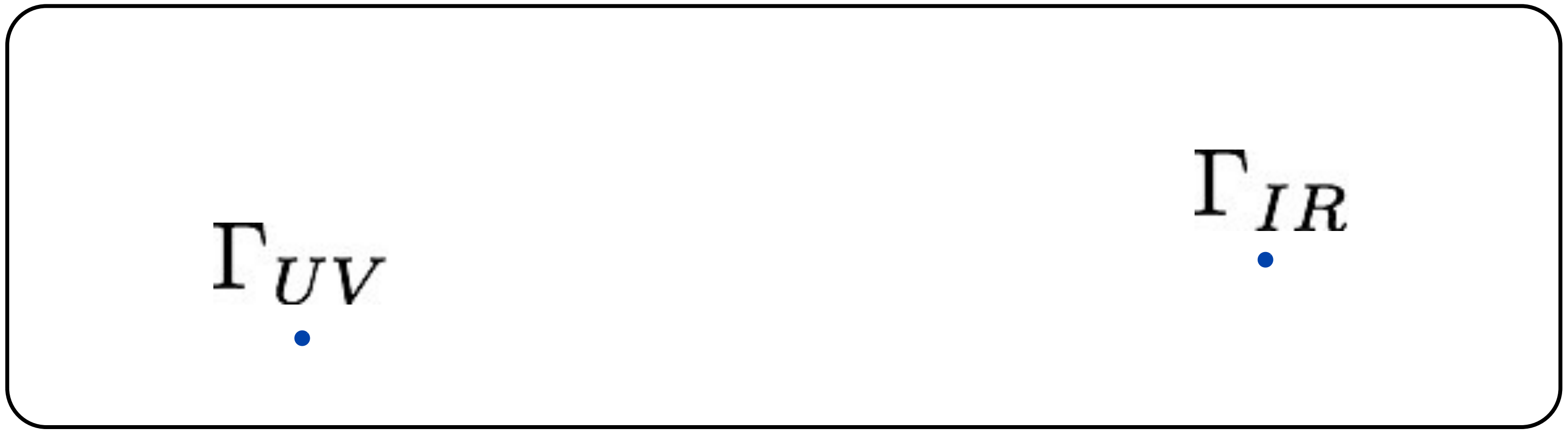


Δc and Δa are universal quantities depending on a whole trajectory!

Strong c- and a-theorems:

$$\partial_t c > 0 \qquad \partial_t a > 0$$

Fixed point action



$$\Gamma_{UV}[\varphi, g] = S_{CFT_{UV}}[\varphi, g] + c_{UV} S_P[g]$$

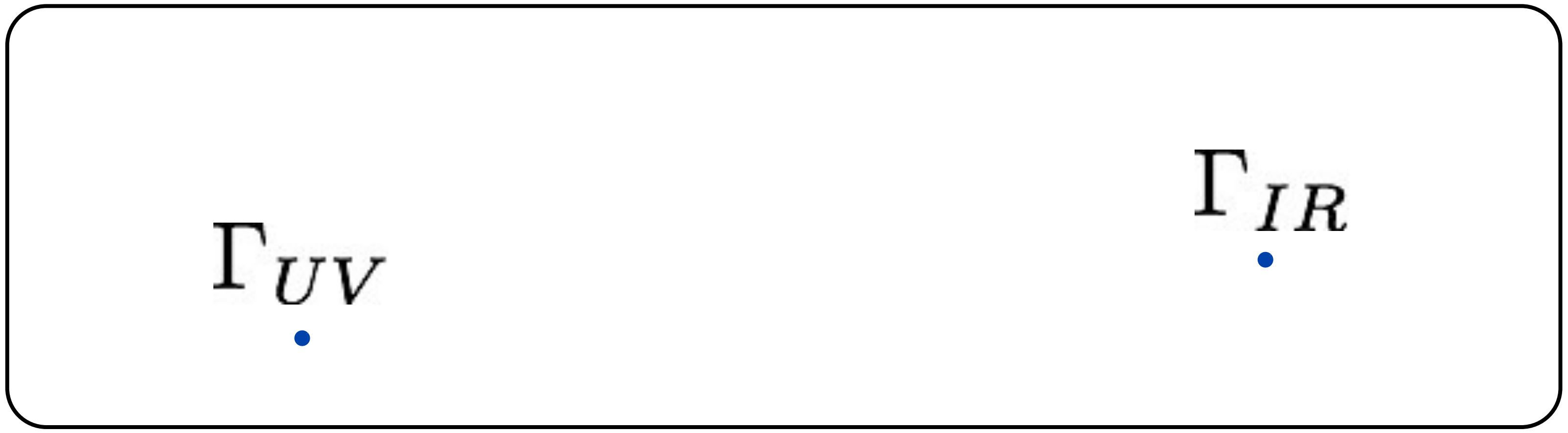
Weyl invariant

Conformal anomaly

$$\langle T^{\mu}_{\mu} \rangle = \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma[g]}{\delta g_{\mu\nu}} \neq 0$$

$$\Gamma_{IR}[\varphi, g] = S_{CFT_{IR}}[\varphi, g] + c_{IR} S_P[g]$$

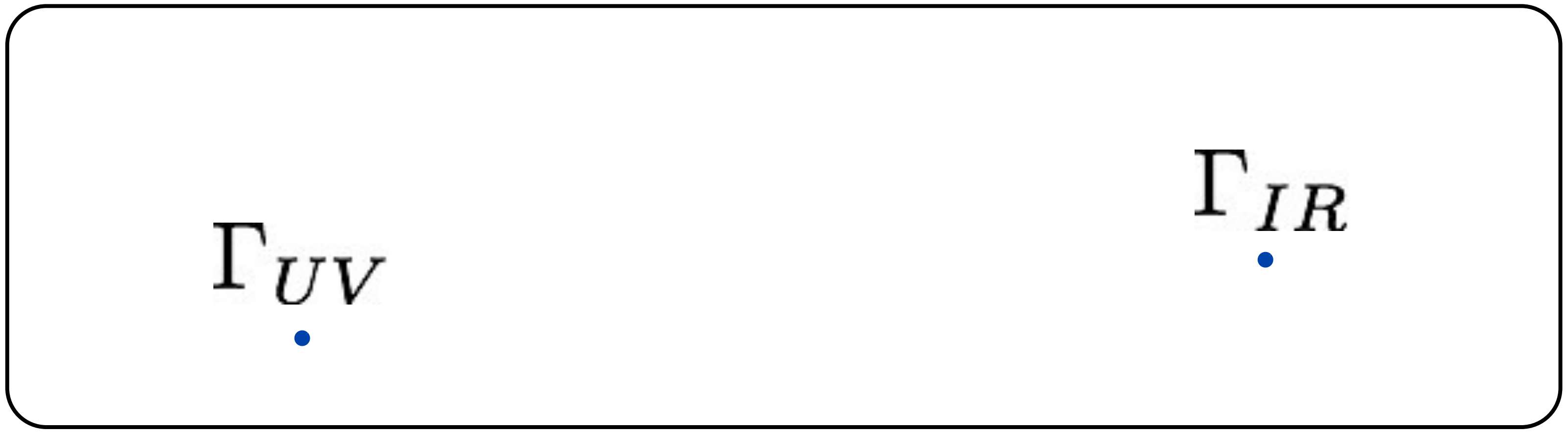
Wess-Zumino action



$$\Gamma[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma[\varphi, g] = \underbrace{S_{CFT}[e^{-w\tau}\varphi, e^{2\tau}g] - S_{CFT}[\varphi, g]}_{=0}$$

$$+\, c(\underbrace{S_R[e^{2\tau}g] - S_R[g])}_{=\Gamma_{WZ}[\tau,g]}$$

Wess-Zumino action



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$$+ c(\underbrace{S_R[e^{2\tau}g] - S_R[g])}_{=\Gamma_{WZ}[\tau,g]}$$

$$\Gamma[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma[\varphi, g] = c\Gamma_{WZ}[\tau, g]$$

Wess-Zumino action

Polyakov & Rigert

d = 2

$$S_P[g] = -\frac{1}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Delta}R$$

$$S_P[e^{2\tau}g] - S_P[g] = \boxed{-\frac{1}{24\pi}\int d^2x\sqrt{g}\,[\tau\Delta\tau + \tau R]}$$

Polyakov & Rigert

d = 2

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$$S_P[e^{2\tau}g] - S_P[g] = \boxed{-\frac{1}{24\pi}\int d^2x\sqrt{g}\,[\tau\Delta\tau + \tau R]}$$

d = 4

$$S_R\left[g\right] = \frac{1}{8}\int d^4x\sqrt{g}\left\{\left[a\left(E+\frac{2}{3}\Delta R\right) - 2cC^2\right]\frac{1}{\Delta_4}\left(E+\frac{2}{3}\Delta R\right)\right\}$$

$$S_R\left[e^{2\tau}g\right] - S_R\left[g\right] = \boxed{-\int d^4x\sqrt{g}\left\{a\left[\left(E+\frac{2}{3}\Delta R\right)\tau + 2\,\tau\Delta_4\tau\right] - cC^2\tau\right\}}$$

Paneitz operator: $\Delta_4 = \Delta^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}\nabla^\mu R\nabla_\mu + \frac{2}{3}R\Delta$

Polyakov & Rigert

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Paneitz operator: $\Delta_4 = \Delta^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}\nabla^\mu R\nabla_\mu + \frac{2}{3}R\Delta$

Getting rid of the dilaton

d = 2

$$\tau(g) = \frac{1}{2\Delta} R \qquad \Gamma_{WZ} \to S_P$$

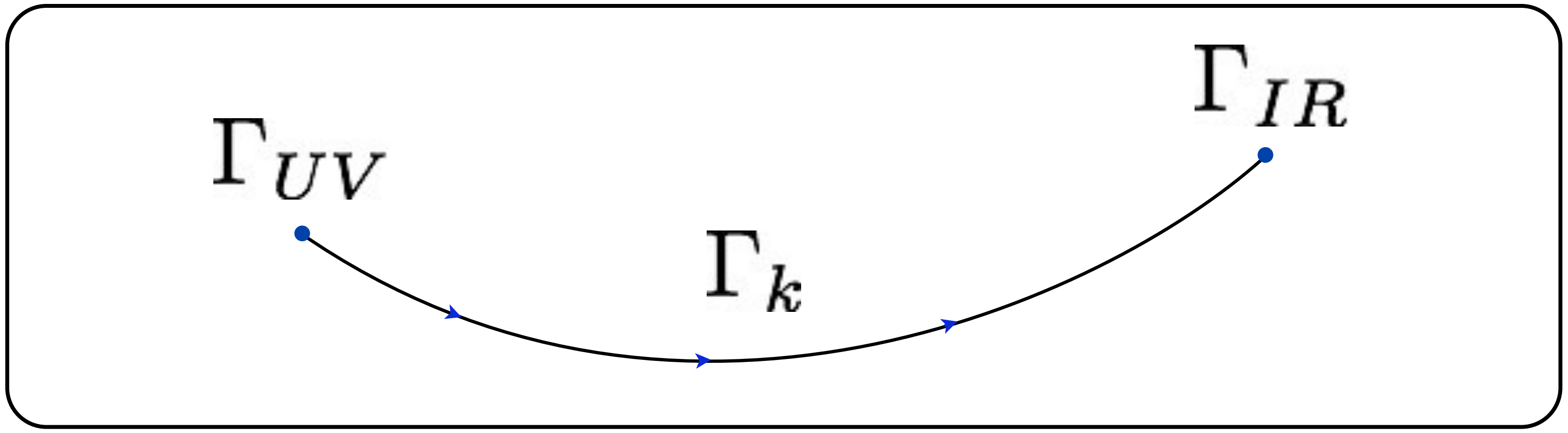
d = 4

$$\tau(g) = -\frac{1}{4\Delta_4}\left(E + \frac{2}{3}\Delta R\right) \qquad \Gamma_{WZ} \to S_R$$

$$\tau(g) = \log\left(1 - \frac{1}{\Delta + \frac{R}{6}} \frac{R}{6}\right)$$

$$\boxed{\tau(e^{2\sigma} g) = \tau(g) + \sigma}$$

Away from the fixed point



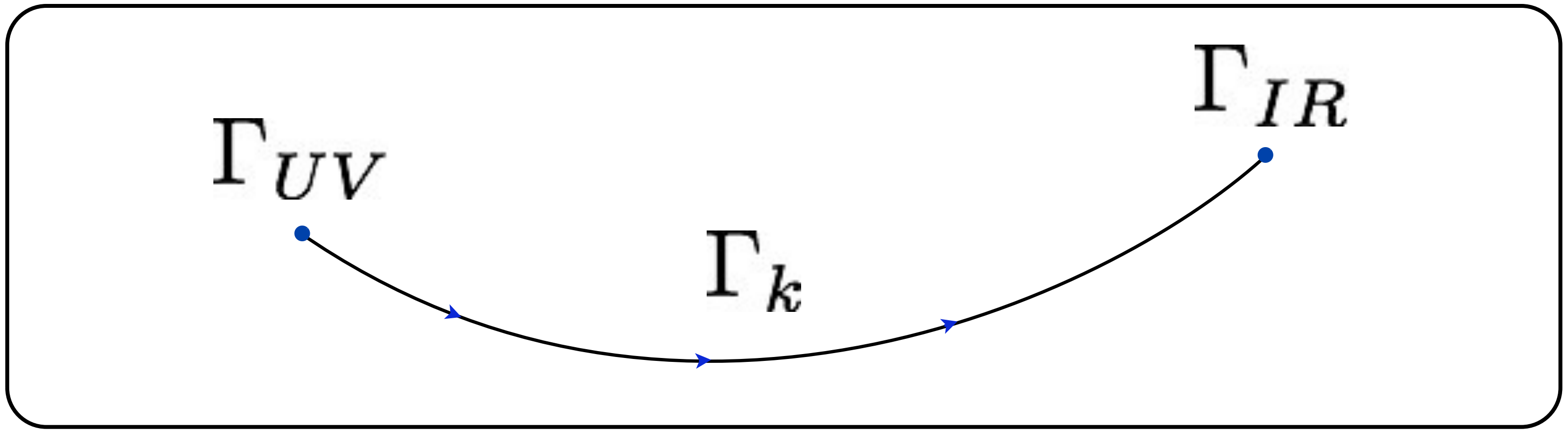
Running Wess-Zumino action:

$$\boxed{\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \mathcal{C}_k \Gamma_{WZ}[\tau, g] + \beta\text{–terms}}$$

Running c-function

Everything that vanishes at a FP...

The integrated c-theorem

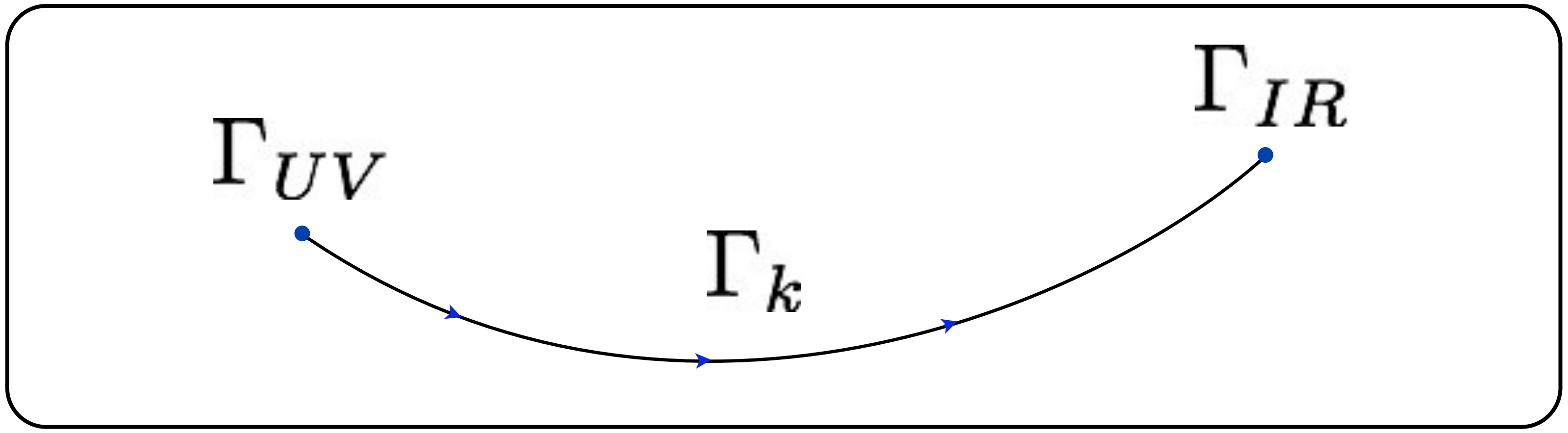


$$\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \mathcal{C}_k \Gamma_{WZ}[\tau, g] + \beta\text{–terms}$$

$\mathcal{C}_k = c_k$ standard quantization

$\mathcal{C}_k = c_k - c_{UV}$ Weyl invariant quantization

The integrated c-theorem



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$\mathcal{C}_k = c_k$ standard quantization

$\mathcal{C}_k = c_k - c_{UV}$ Weyl invariant quantization

$$\Gamma_{IR}[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_{IR}[\varphi, g] = (c_{IR} - c_{UV})\Gamma_{WZ}[\tau, g]$$

The integrated c-theorem from the EAA

*

$$\Gamma_{IR}[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_{IR}[\varphi, g] = (c_{IR} - c_{UV})\Gamma_{WZ}[\tau, g]$$

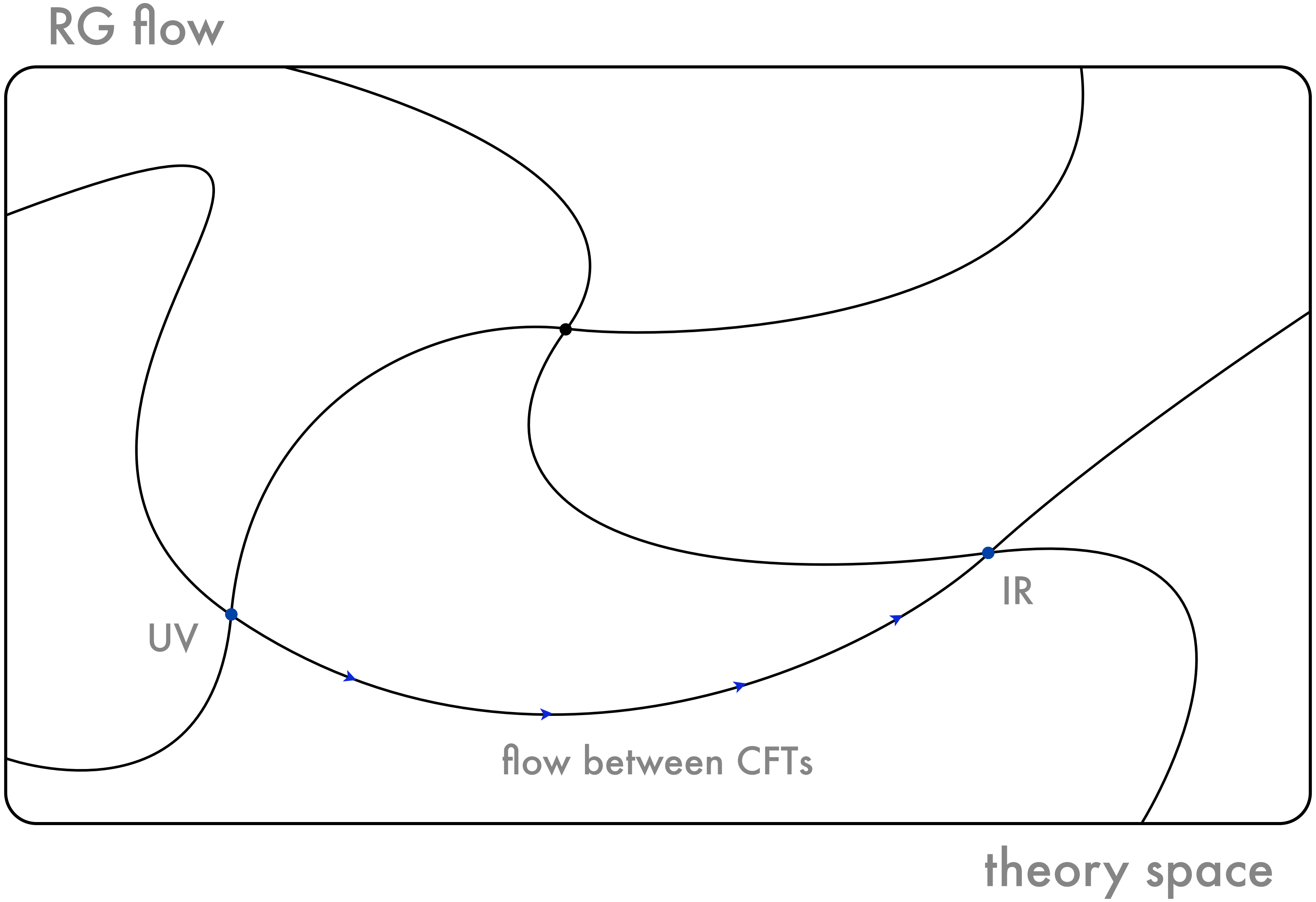
$$\Gamma_{IR}[e^{-w\tau}\varphi, e^{2\tau}g] = \Gamma_{IR}[\varphi, g] + \int \sqrt{g}\,\tau_x \frac{\delta}{\delta\tau_x}\Gamma_{IR}[e^{-w\tau}\varphi, e^{2\tau}g]\Big|_{\tau\to 0}$$
$$+\frac{1}{2}\int_x \sqrt{g_x}\int_y \sqrt{g_y}\,\tau_x\tau_y \frac{\delta^2}{\delta\tau_x\delta\tau_y}\Gamma_{IR}[e^{-w\tau}\varphi, e^{2\tau}g]\Big|_{\tau\to 0} + \mathcal{O}\left(\tau^3\right)$$

$$\frac{1}{4}\int_{xy} \tau_x (y-x)^\mu (y-x)^\nu \partial_\mu\partial_\nu\tau_x \left\langle \Theta_x\Theta_y \right\rangle_{IR} = \frac{\Delta c}{24\pi}\int_x \tau_x \Delta\tau_x$$

$$\boxed{\Delta c = 3\pi \int d^2x\, x^2 \left\langle \Theta(x)\Theta(0)\right\rangle_{IR}}$$

The integrated c-theorem

*



$$\Delta c = c_{UV} - c_{IR} = 3\pi \int d^2x\, x^2 \ \langle \Theta(x) \Theta(0) \rangle \geq 0$$

The flow of the c-function

$$\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \mathcal{C}_k \Gamma_{WZ}[\tau, g] + \beta\text{–terms}$$

The flow of the c-function

$$\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \mathcal{C}_k\Gamma_{WZ}[\tau, g] + \beta\text{–terms}$$

$$\boxed{\partial_t c_k = -24\pi\,\partial_t\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g]\Big|_{\int\tau\Delta\tau}}$$

Exact flow for the c-function!

The flow of the c-function

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Exact flow for the c-function!

$$\partial_t c_k = -12\pi \frac{\delta^2}{\delta\tau_p \delta\tau_{-p}} \operatorname{Tr}\left(\frac{\partial_t R_k[\tau]}{\Gamma_k^{(2;0)}[\varphi;\tau] + R_k[\tau]}\right)\Bigg|_{p^2}$$

The flow of the c-function

$$\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \mathcal{C}_k \Gamma_{WZ}[\tau, g] + \beta\text{–terms}$$

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Exact flow for the c-function!

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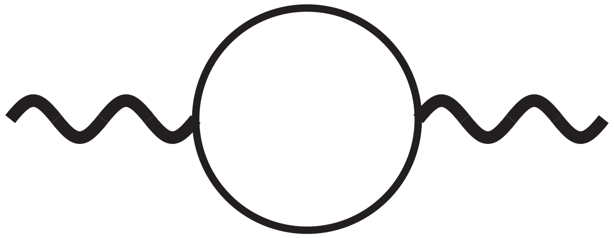
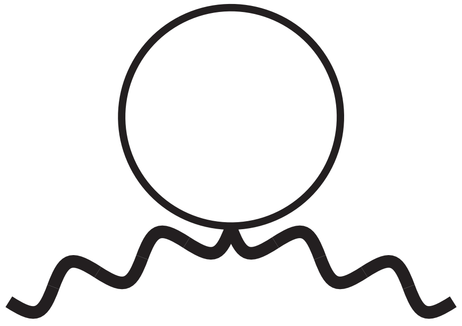
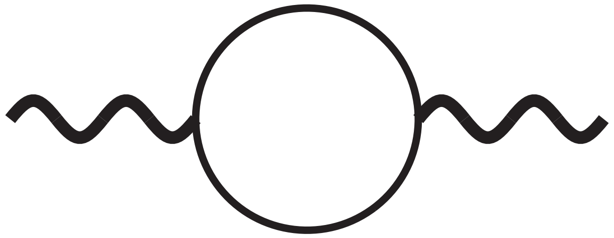
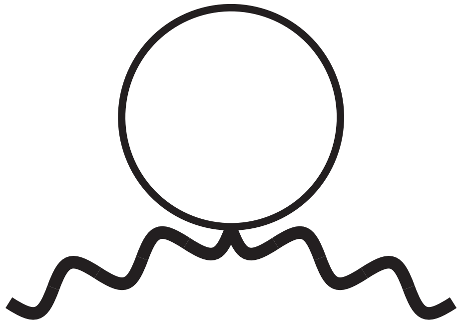
Similarly for the a-function in d = 4

The flow of the c-function

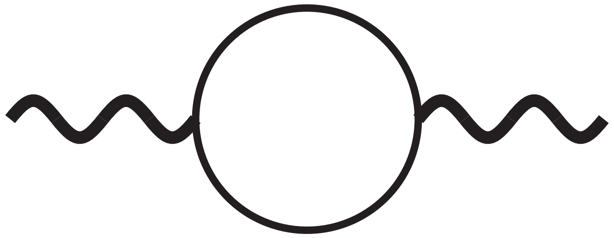
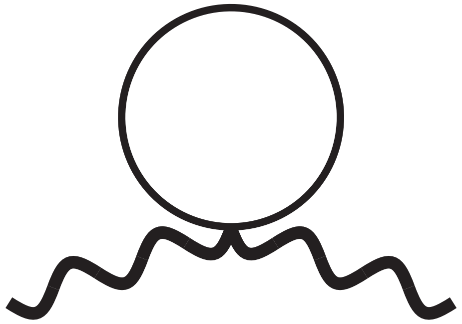
$\partial_t c_k = 12\pi$  -12π  $\Big|_{p^2}$

The flow of the c-function

$\partial_t c_k = 12\pi$  -12π  $\Big|_{p^2}$

The flow is driven by matter-dilaton interactions...

The flow of the c-function

$\partial_t c_k = 12\pi$  -12π  $\Big|_{p^2}$

$$\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \mathcal{C}_k \Gamma_{WZ}[\tau, g] + \beta\text{–terms}$$

The flow is driven by matter-dilaton interactions…

Which is the nature of the β–terms ?

Running WZ action: $\Gamma_k[\tau, g]$

$$\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \boxed{\mathcal{C}_k \Gamma_{WZ}[\varphi, \tau; g] + \beta\text{–terms}}$$

Relevant (primary) perturbations

$$\boxed{\Gamma_k}$$

$$\Gamma_k[\varphi, g] = \Gamma_{UV}[\varphi, g] + \sum_i g^i \int \sqrt{g}\, \mathcal{O}_i[\varphi, g]$$

Coupling constants

What is the general form
of the effective action
away from criticality?

Which is the nature of the β–terms ?

Running WZ action: $\Gamma_k[\tau, g]$

$$\Gamma_k[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \boxed{\mathcal{C}_k\Gamma_{WZ}[\varphi, \tau; g] + \beta\text{–terms}}$$

$\boxed{\Gamma_k}$

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Relevant (primary) perturbations

$$\Gamma_k[\varphi, g] = \Gamma_{UV}[\varphi, g] + \sum_i g^i \int \sqrt{g}\, \mathcal{O}_i[\varphi, g]$$

Coupling constants

Clue I: conformal anomaly
Clue II: scale anomaly
Clue III: Stuckelberg trick & Local RG

Scale anomaly

Scale anomaly (classical + quantum):

$$\int \sqrt{g}\,\langle T^\mu_\mu \rangle = -\sum_i (\underset{\text{quantum}}{\beta^i} - \underset{\text{classical}}{d_i g^i}) \int \sqrt{g}\mathcal{O}_i$$

Dimensionless couplings and beta functions:

$$g^i = k^{d_i}\,\tilde{g}^i \qquad \beta^i - d_i g^i = k^{d_i}\,\tilde{\beta}^i$$

$$\beta\text{–terms} = -\sum_i k^{d_i}\,\tilde{\beta}^i \int \sqrt{g}\,\tau\mathcal{O}_i + \ldots$$

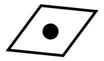
First interaction contribution to the flow of c and a!

Scale anomaly (example)

$*$

$$S = \int \sqrt{g}\left[\frac{1}{2}\phi\Delta\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\right]$$

$d = 4$



Gaussian fixed point

"classical" scale anomaly:

$$\delta_\sigma S = 2m^2 \int \sqrt{g}\frac{1}{2}\phi^2\sigma$$

Scale anomaly (example)

$*$

$$S = \int \sqrt{g} \left[\frac{1}{2} \phi \Delta \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right]$$

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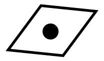

Gaussian fixed point

"classical" scale anomaly:

$$\delta_\sigma S = 2m^2 \int \sqrt{g} \frac{1}{2} \phi^2 \sigma$$

quantum scale anomaly:

$$\delta_\sigma \Gamma_{1\text{–loop}} = -\beta_m(m_R^2) \int \sqrt{g} \frac{1}{2} \phi^2 \sigma - \beta_\lambda(\lambda_R) \int \sqrt{g} \frac{1}{4!} \phi^4 \sigma$$

Scale anomaly (example)

*

$$S = \int \sqrt{g} \left[\frac{1}{2}\phi\Delta\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \right]$$

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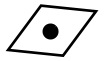

Gaussian fixed point

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$$\beta_m = \frac{m^2\lambda}{(4\pi)^2}$$

$$\beta_\lambda = \frac{3\lambda^2}{(4\pi)^2}$$

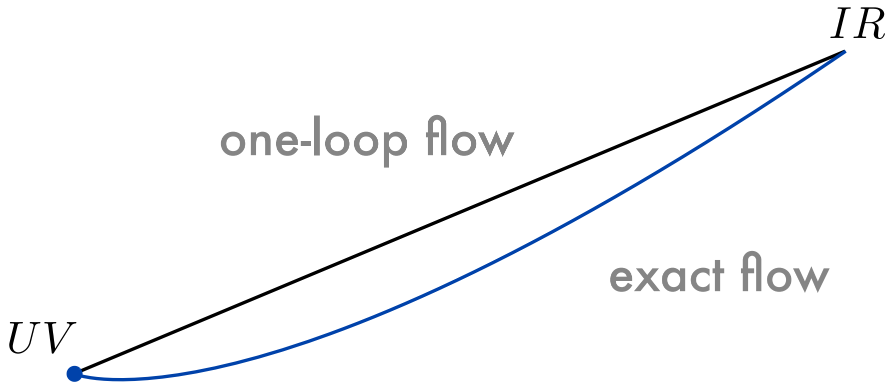
Scale anomaly (example)

$*$

$d = 4$

$$\delta_\sigma \Gamma = \delta_\sigma S + \delta_\sigma \Gamma_{1-\text{loop}}$$

$$= -\left(\beta_m(m_R^2) - 2m_R^2\right) \int \sqrt{g} \frac{1}{2} \phi^2 \sigma - \beta_\lambda(\lambda_R) \int \sqrt{g} \frac{1}{4!} \phi^4 \sigma$$



quantum energy-momentum tensor:

$$\delta_\sigma \Gamma = \int \sqrt{g} \left\langle T^\mu_\mu \right\rangle \sigma$$

$$\left\langle T^\mu_\mu \right\rangle = -\left(\beta_m(m_R^2) - 2m_R^2\right) \int \sqrt{g} \frac{1}{2} \phi^2 - \beta_\lambda(\lambda_R) \int \sqrt{g} \frac{1}{4!} \phi^4$$

A derivative expansion for the β–terms

$$\beta\text{–terms} = \int \sqrt{g}\,\{-\tau\beta^i\mathcal{O}_i + O(\tau^2)\}$$

$$\Gamma_k[\tau, g] = \int \sqrt{g}\,[V_k(\tau) + Z_k(\tau)\partial_\mu\tau\partial^\mu\tau + F_k(\tau)R] + O(\partial^4)$$

$$V_k(\tau) = -\tau\beta^i\mathcal{O}_i + ...$$
$$Z_k(\tau) = -\frac{\mathcal{C}_k}{24\pi} + ...$$
$$F_k(\tau) = -\frac{\mathcal{C}_k}{24\pi}\tau + ...$$

How do we determine the next terms?

Stuckelberg trick & Local RG

Stuckelberg trick:

$$k \to e^{-\tau} k$$

Couplings become spacetime dependent!

$$g^i_k \to g^i_{ke^{-\tau}}$$

Natural way to introduce beta functions:

$$\begin{aligned} g^i_{ke^{-\tau}} &= g^i_{k(1-\tau+\ldots)} \\ &= g^i_k - \tau k \partial_k g^i_k + \ldots \\ &= g^i_k - \tau \beta^i_k + \ldots \end{aligned}$$

Physical idea: Weyl transformations can be compensated by RG rescalings...

Stuckelberg trick & Local RG

The CFT actions delete each other

Recovering the scale anomaly:

Marginal deformation

$$\Gamma_{e^{-\tau}k}[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \int \sqrt{g}\left[(g^i_k - \tau\beta^i_k)\mathcal{O}_i - g^i_k\mathcal{O}_i\right]$$

$$= -\int \sqrt{g}\tau\beta^i_k\mathcal{O}_i$$

$$\equiv \int \sqrt{g}\tau\left\langle\Theta\right\rangle$$

$$\langle\Theta\rangle = -\beta^i_k\mathcal{O}_i$$

Stuckelberg trick & Local RG

The CFT actions delete each others

Recovering the scale anomaly:

Marginal deformation

$$\Gamma_{e^{-\tau}k}[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \int \sqrt{g}\left[(g^i_k - \tau\beta^i_k)\mathcal{O}_i - g^i_k\mathcal{O}_i\right]$$

$$= -\int \sqrt{g}\tau\beta^i_k\mathcal{O}_i$$

$$\langle\Theta\rangle = -\beta^i_k\mathcal{O}_i$$

$$\equiv \int \sqrt{g}\tau\left\langle\Theta\right\rangle$$

Osborn's terms:

All possible dimensionless terms

$$\beta\text{–terms} = \int \sqrt{g}\left[-\tau\beta^i\mathcal{O}_i + \chi_{ij}\partial_\mu g^i\partial^\mu g^j\tau + \omega_i\partial_\mu\tau\partial^\mu g^i\right]$$

Stuckelberg trick & Local RG

The CFT actions delete each others

Recovering the scale anomaly:

Marginal deformation

$$\Gamma_{e^{-\tau}k}[e^{-w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \int \sqrt{g}\left[(g^i_k - \tau\beta^i_k)\mathcal{O}_i - g^i_k\mathcal{O}_i\right]$$

$$= -\int \sqrt{g}\tau\beta^i_k\mathcal{O}_i$$

$$\langle\Theta\rangle = -\beta^i_k\mathcal{O}_i$$

$$\equiv \int \sqrt{g}\tau\,\langle\Theta\rangle$$

Osborn's terms:

All possible dimensionless terms

$$\beta\text{–terms} = \int \sqrt{g}\left[-\tau\beta^i\mathcal{O}_i + \chi_{ij}\partial_\mu g^i\partial^\mu g^j\tau + \omega_i\partial_\mu\tau\partial^\mu g^i\right]$$

Spacetime dependent couplings play the role of currents...

Stuckelberg trick & Local RG

Non-trivial relations:

$$g^i_{ke^{-\tau}} = g^i_k - \tau\beta^i_k + \frac{1}{2}\tau^2\beta^j\partial_j\beta^i + O(\tau^3)$$

Reduce to dilaton interactions:

$$\begin{aligned}
\partial_\mu g^i &= -\beta^i\partial_\mu\tau + O(\tau^2) \\
\chi_{ij}\partial_\mu g^i\partial^\mu g^j &= \chi_{ij}\beta^i\beta^j\partial_\mu\tau\partial^\mu\tau + O(\tau^3) \\
\omega_i\partial_\mu\tau\partial^\mu g^i &= -\omega_i\beta^i\partial_\mu\tau\partial^\mu\tau \\
&\quad + \partial_t\left(\omega_i\beta^i\right)\tau\partial_\mu\tau\partial^\mu\tau + O(\tau^4)
\end{aligned}$$

Apply to the c-function:

$$\mathcal{C}_{ke^{-\tau}} = \mathcal{C}_k - \tau\partial_t\mathcal{C}_k + O(\tau^2)$$

Stuckelberg trick & Local RG

$$V_k(\tau) = -\beta^i \mathcal{O}_i \tau + \frac{1}{2}\beta^j \partial_j \beta^i \mathcal{O}_i \tau^2 + O(\tau^3)$$

$$\begin{aligned} Z_k(\tau) &= -\frac{\mathcal{C}_k}{24\pi} - \omega_i \beta^i \\ &\quad + \left[-\partial_t \left(\frac{\mathcal{C}_k}{24\pi} + \omega_i \beta^i\right) + \chi_{ij}\beta^i \beta^j\right]\tau + O(\tau^2) \end{aligned}$$

$$F_k(\tau) = -\frac{\mathcal{C}_k}{24\pi}\tau + \partial_t \left(\frac{\mathcal{C}_k}{24\pi}\right)\tau^2 + O(\tau^3)$$

$$\boxed{c_k = \mathcal{C}_k + 24\pi\omega_i \beta^i}$$

Wess-Zumino consistency conditions

$$\Gamma[e^{2\tau}g] - \Gamma[g] = \Gamma[\tau, g]$$

Weyl transformations are Abelian:

$$\Gamma[e^{2\tau_2}e^{2\tau_1}g] - \Gamma[e^{2\tau_1}g] = \Gamma[\tau_1, e^{2\tau_2}g]$$
$$\Gamma[e^{2\tau_1}e^{2\tau_2}g] - \Gamma[e^{2\tau_2}g] = \Gamma[\tau_2, e^{2\tau_1}g]$$

$$\downarrow$$

$$\Gamma[\tau_1, e^{2\tau_2}g] - \Gamma[\tau_1, g] = \Gamma[\tau_2, e^{2\tau_1}g] - \Gamma[\tau_2, g]$$

Infinitesimal WZ consistency conditions:

$$\Gamma[\tau_1, e^{2\tau_2}g] = \Gamma[\tau_1, g] + \delta_{\tau_2}\Gamma[\tau_1, g] + \ldots$$

$$\boxed{\delta_{\tau_2}\Gamma[\tau_1, g] = \delta_{\tau_1}\Gamma[\tau_2, g]}$$

Wess-Zumino consistency conditions

Consistency condition deriving from the terms $\tau\partial_\mu\tau\partial^\mu\tau$

$$-\partial_t\left(\frac{\mathcal{C}_k}{24\pi}+\omega_i\beta^i\right)+\chi_{ij}\beta^i\beta^j=0$$

$$\boxed{\partial_t c_k=24\pi\chi_{ij}\beta^i\beta^j}$$

Wess-Zumino consistency conditions

Consistency condition deriving from the terms $\tau\partial_\mu\tau\partial^\mu\tau$

↓

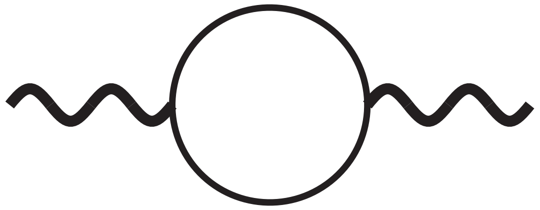
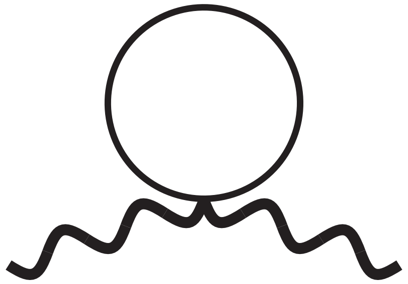
$$-\partial_t\left(\frac{\mathcal{C}_k}{24\pi}+\omega_i\beta^i\right)+\chi_{ij}\beta^i\beta^j=0$$

↓

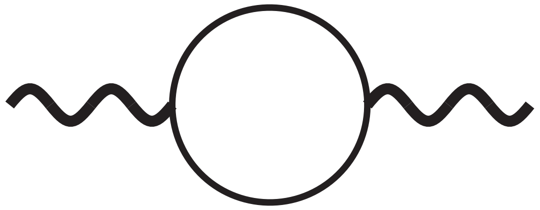
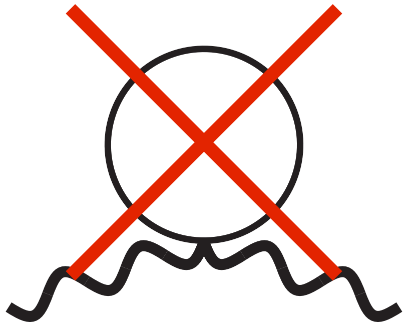
$$\boxed{\partial_t c_k = 24\pi\chi_{ij}\beta^i\beta^j}$$

Consistency conditions don't tell us how to compute things...

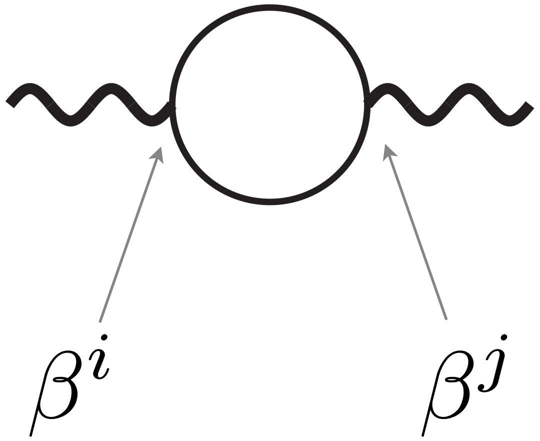
Zamolodchikov's metric

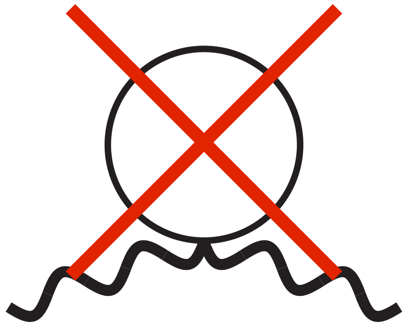
$\partial_t c_k \; = \; 12\pi$  -12π  $\Big|_{p^2}$

Zamolodchikov’s metric

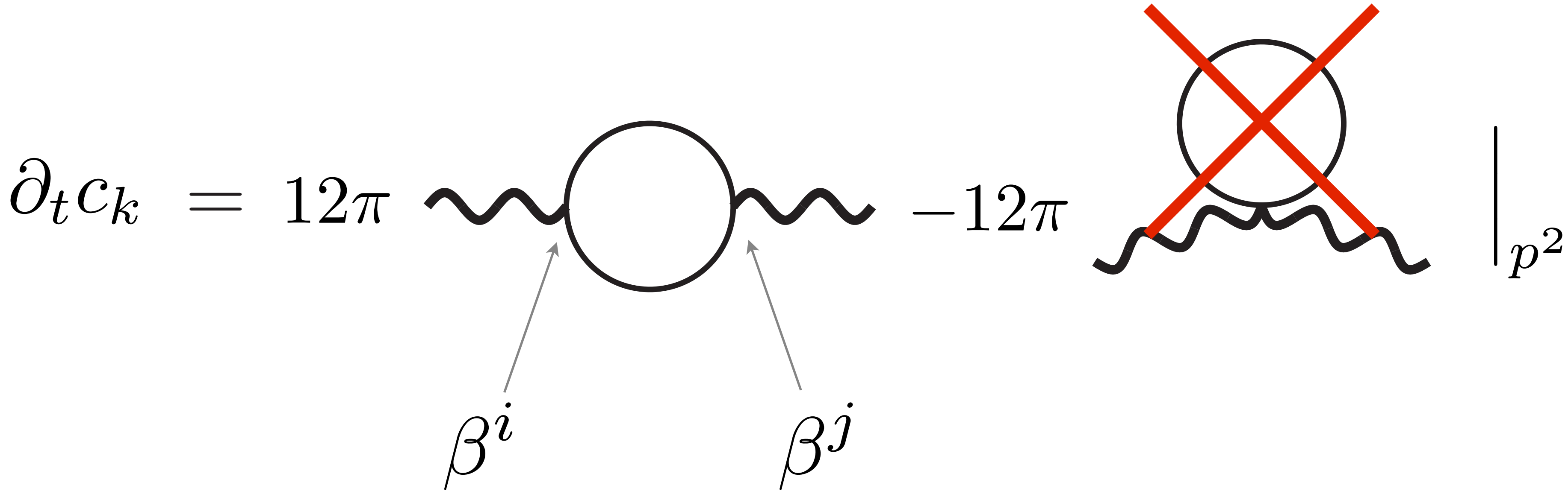
$\partial_t c_k \;=\; 12\pi$  -12π  $\Big|_{p^2}$

Zamolodchikov's metric

$\partial_t c_k \;=\; 12\pi$ 

-12π  $\Big|_{p^2}$

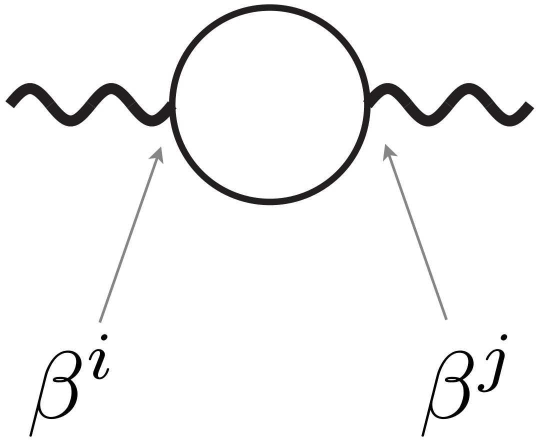
Zamolodchikov's metric



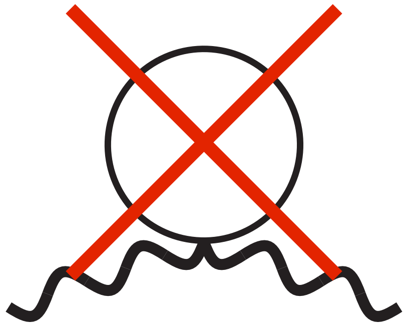
$$\partial_t c_k = 24\pi \chi_{ij} \beta^i \beta^j$$

Zamolodchikov's metric

$$\partial_t c_k = 12\pi$$



$$-12\pi$$



$$\Big|_{p^2}$$

$$\partial_t c_k = 24\pi \chi_{ij} \beta^i \beta^j$$

$$\boxed{\chi_{ij} = \frac{1}{24\pi} \int \frac{d^2 q}{(2\pi)^2} \tilde{\partial}_t \left\{ G_k(q^2) G_k\left((q+p)^2\right) \right\} \mathcal{O}_i \mathcal{O}_j}$$

Massive deformation Gaussian FP

$$\Gamma_k[\phi, g] = \frac{1}{2}\int \sqrt{g}\phi\left(\Delta + m^2\right)\phi - \frac{c_k}{96\pi}\int \sqrt{g}R\frac{1}{\Delta}R$$

$$\Gamma_k[\phi, e^{2\tau}\delta] = \frac{1}{2}\int \phi\left(\Delta + e^{2\tau}m^2\right)\phi - \frac{c_k}{24\pi}\int \tau\Delta\tau$$

Massive deformation Gaussian FP

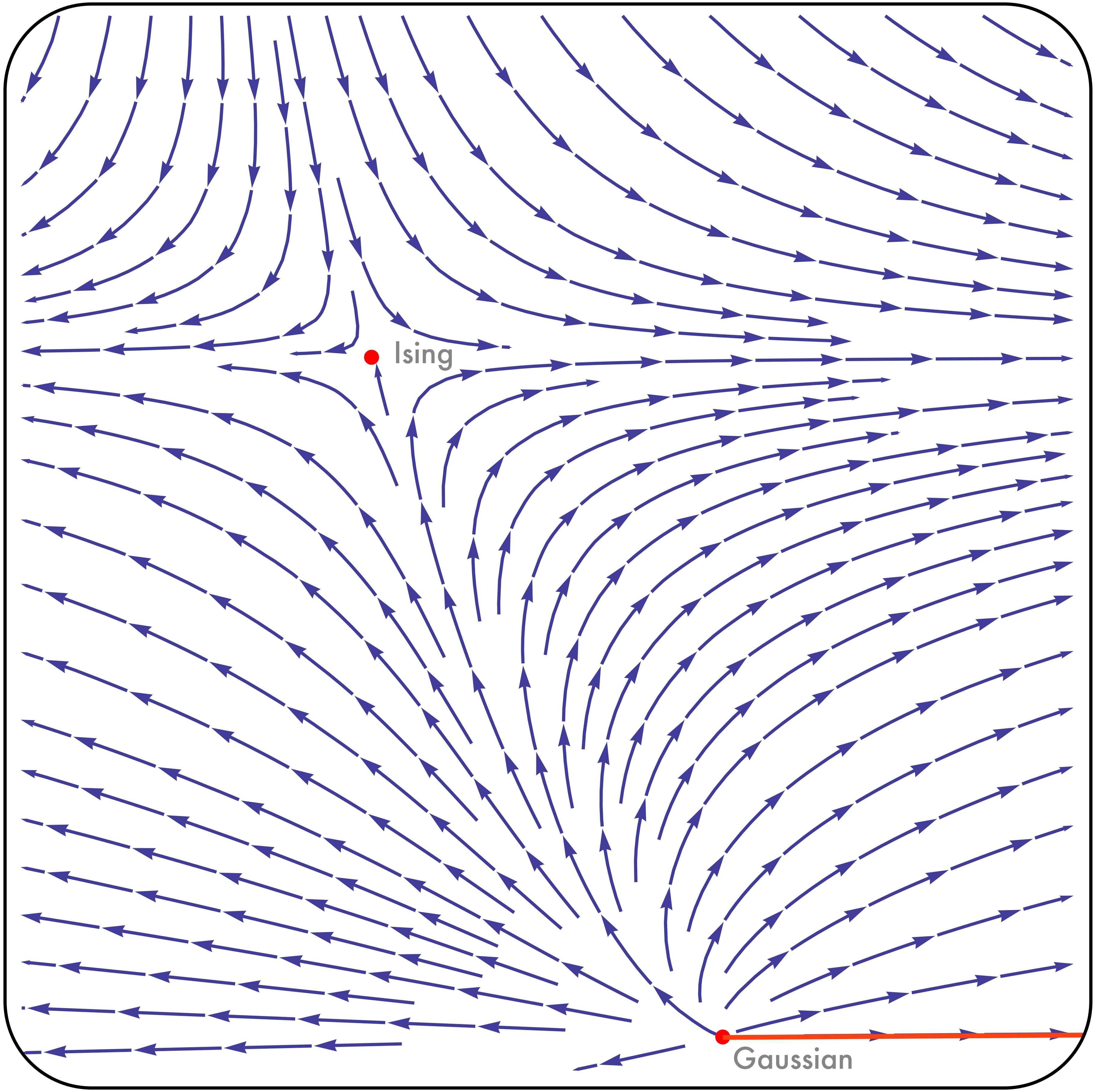
$$\Gamma_k[\phi,g]=\frac{1}{2}\int\sqrt{g}\phi\left(\Delta+m^2\right)\phi-\frac{c_k}{96\pi}\int\sqrt{g}R\frac{1}{\Delta}R$$

$$\Gamma_k[\phi,e^{2\tau}\delta]=\frac{1}{2}\int\phi\left(\Delta+e^{2\tau}m^2\right)\phi-\frac{c_k}{24\pi}\int\tau\Delta\tau$$

$$\partial_t c_k=\frac{4ak^2m^4}{\left(ak^2+m^2\right)^3}\qquad\qquad R_k(z)=ak^2$$

$$c_k=1-\frac{m^4}{\left(ak^2+m^2\right)^2}$$

$$c_\infty=1\qquad\qquad c_0=0\qquad\qquad \Delta c=1$$



theory space

Massive deformation Wilson-Fisher FP

$$\Gamma_k[\bar{\psi},\psi,g]=\int\sqrt{g}\,\bar{\psi}\,(\nabla+m)\,\psi-\frac{c_k}{96\pi}\int\sqrt{g}R\frac{1}{\Delta}R$$

$$\Gamma_k[e^{\tau/2}\bar{\psi},e^{\tau/2}\psi,e^{2\tau}\delta]=\int\bar{\psi}\,(\nabla+e^{\sigma}m)\,\psi-\frac{c_k}{24\pi}\int\tau\Delta\tau$$

Massive deformation Wilson-Fisher FP

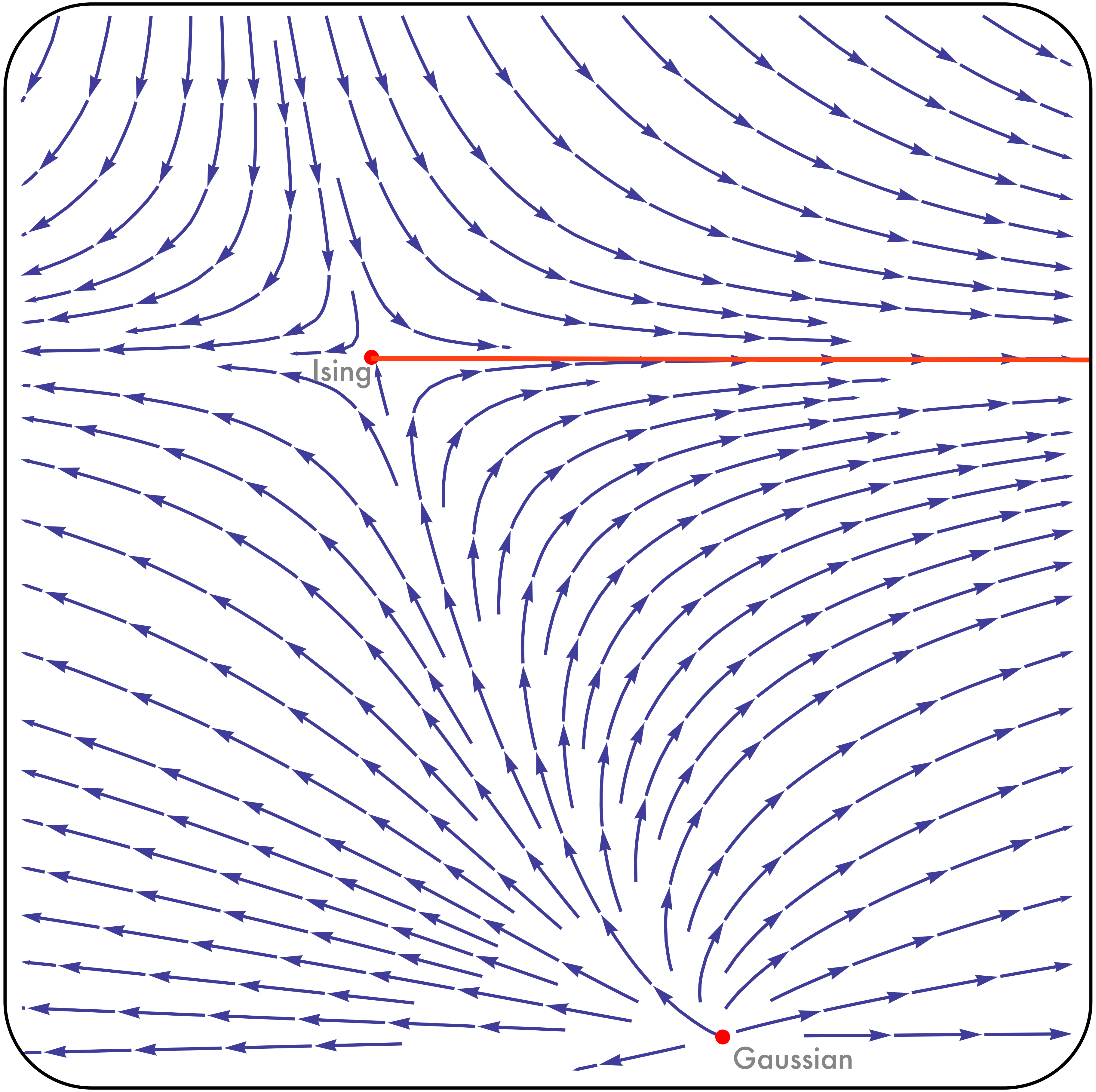
$$\Gamma_k[\bar{\psi}, \psi, g] = \int \sqrt{g}\, \bar{\psi}\, (\nabla + m)\, \psi - \frac{c_k}{96\pi} \int \sqrt{g} R \frac{1}{\Delta} R$$

$$\Gamma_k[e^{\tau/2}\bar{\psi}, e^{\tau/2}\psi, e^{2\tau}\delta] = \int \bar{\psi}\, (\nabla + e^{\sigma} m)\, \psi - \frac{c_k}{24\pi} \int \tau \Delta \tau$$

$$\partial_t c_k = \frac{akm^2}{(ak+m)^3}$$

$$c_k = \frac{1}{2} - \frac{m^2}{2\,(ak+m)^2}$$

$$c_\infty = \frac{1}{2} \qquad c_0 = 0 \qquad \Delta c = \frac{1}{2}$$



theory space

The *c*-function in the LPA

extend a given truncation:

$$\Gamma_k[\varphi] = \int \left[V_k(\varphi) + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + ... \right]$$

↓

$$\begin{aligned}\Gamma_k[\varphi, g] = \int \sqrt{g} \Big[& V_k(\varphi) + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + ... \\ & - \frac{1}{2}\partial_t V_k(\varphi)\frac{1}{\Delta}R + ... \\ & - \frac{c_k - c_\Lambda}{96\pi} R\frac{1}{\Delta}R + ... \Big]\end{aligned}$$

The *c*-function in the LPA

non-perturbative flow for the c-function:

$$\partial_t c_k = -24\pi\,\partial_t \Gamma_k[e^{-w\tau}\varphi, e^{2\tau} g]\Big|_{\int \tau \Delta \tau}$$

The *c*-function in the LPA

non-perturbative flow for the c-function:

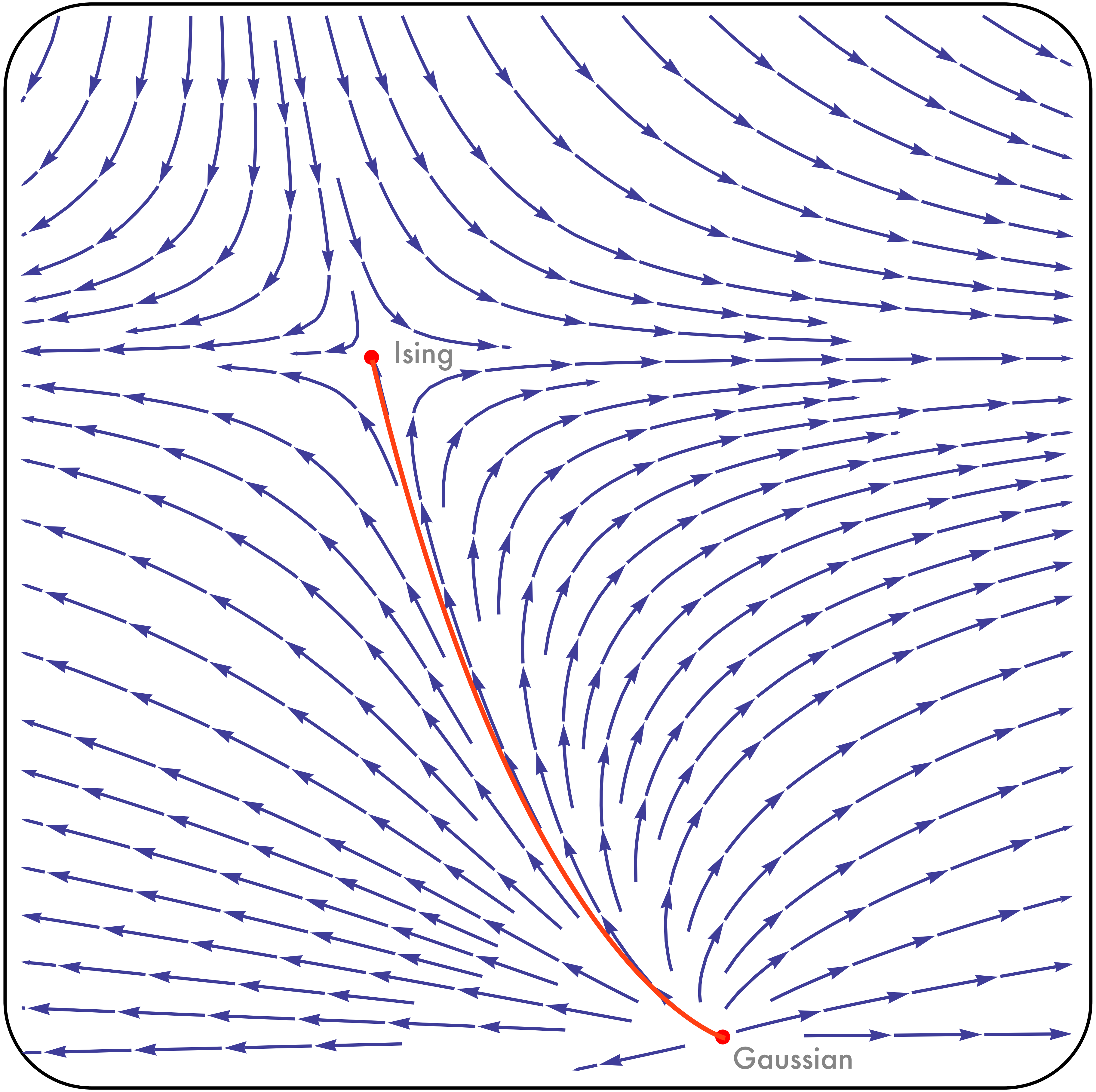
$$\partial_t c_k = -24\pi\, \partial_t \Gamma_k[e^{-w\tau}\varphi, e^{2\tau} g]\Big|_{\int \tau \Delta \tau}$$

the c-function with in the LPA:

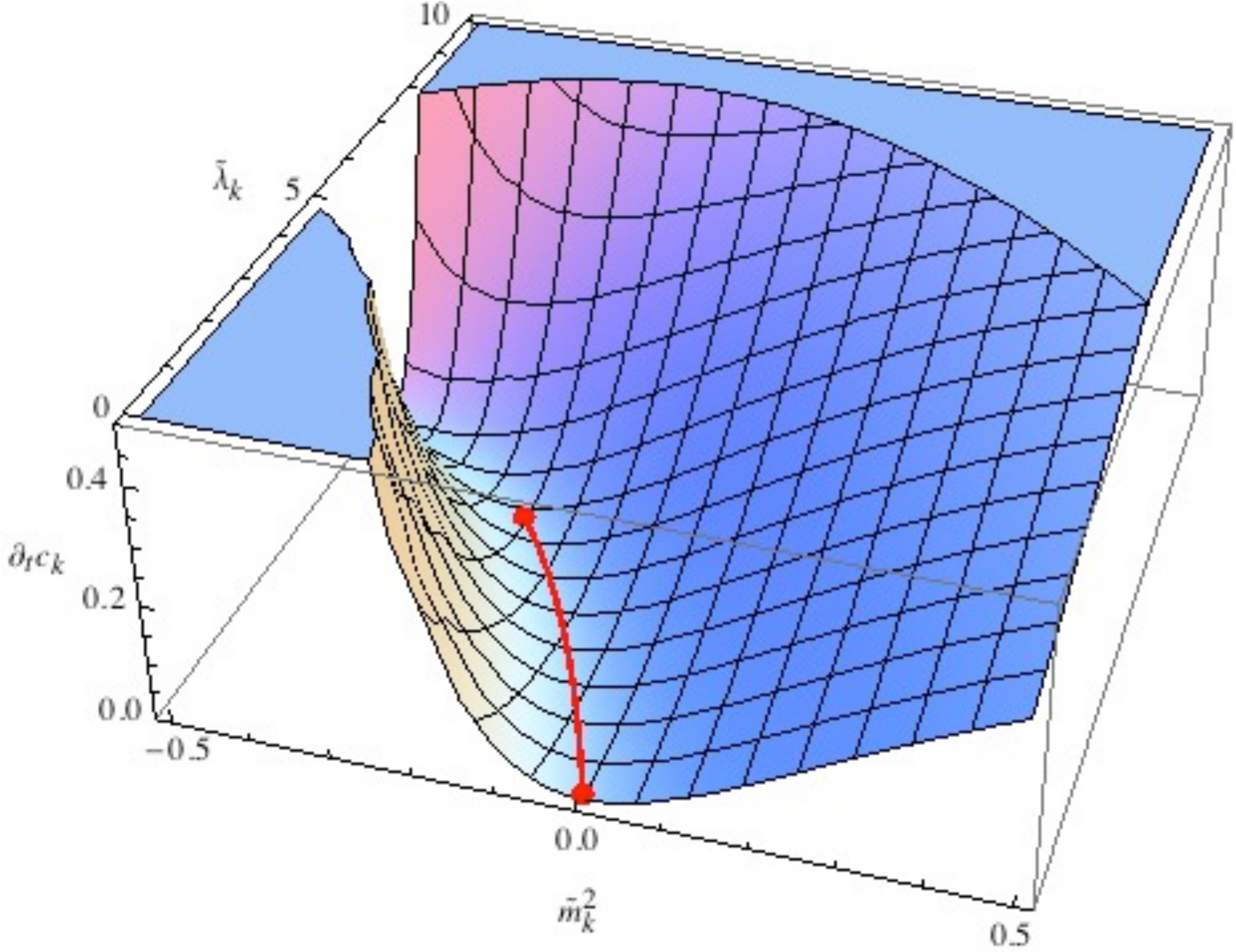
$$\begin{aligned}\partial_t c_k &= \frac{12}{(1+\tilde{m}_k^2)^4}\left(\tilde{\beta}_{m^2}\right)^2 \\ &= \frac{12}{(1+\tilde{m}_k^2)^4}\left(2\tilde{m}_k^2 + \frac{1}{4\pi}\frac{\tilde{\lambda}_k}{(1+\tilde{m}_k^2)^2}\right)^2\end{aligned}$$

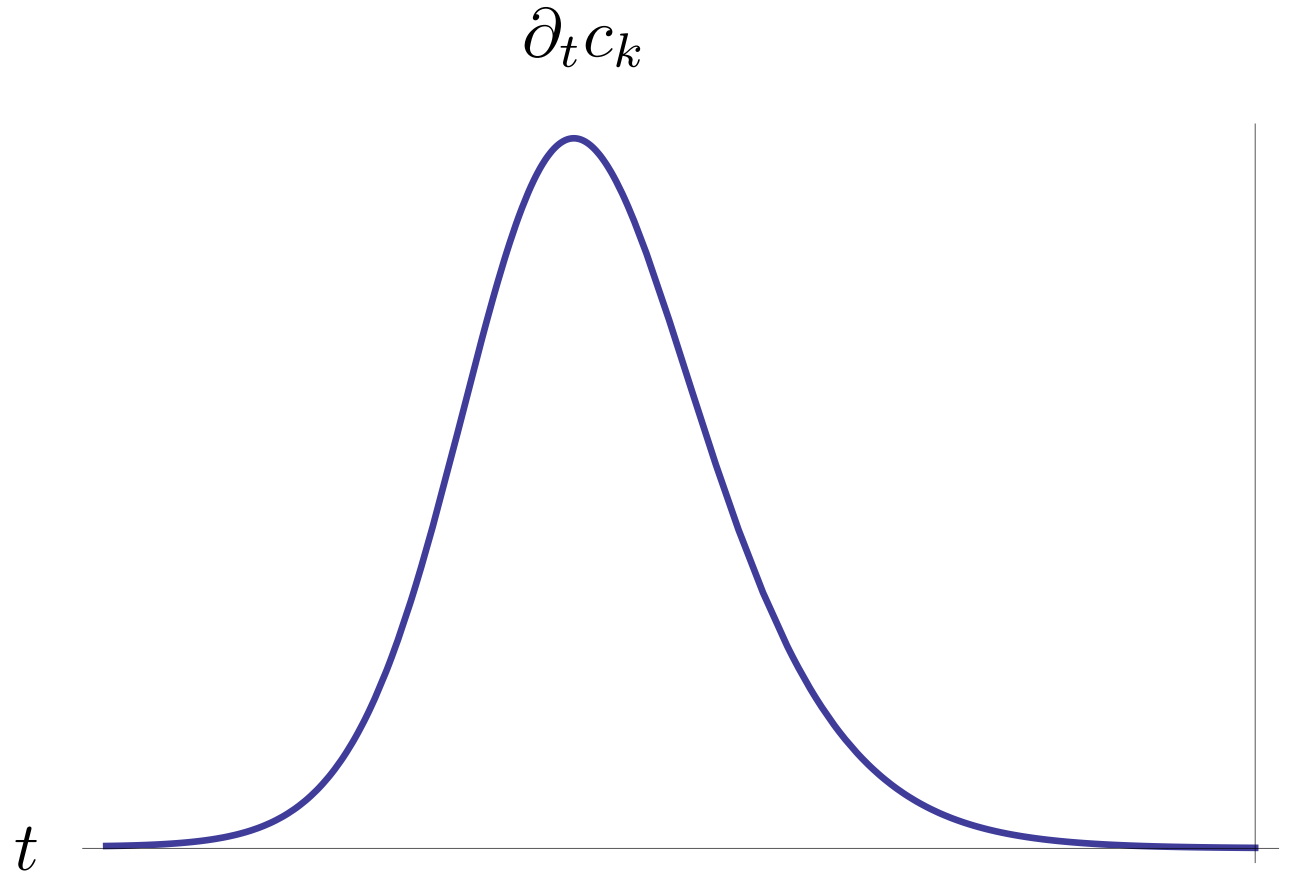
the c-theorem is satisfied within our truncation!

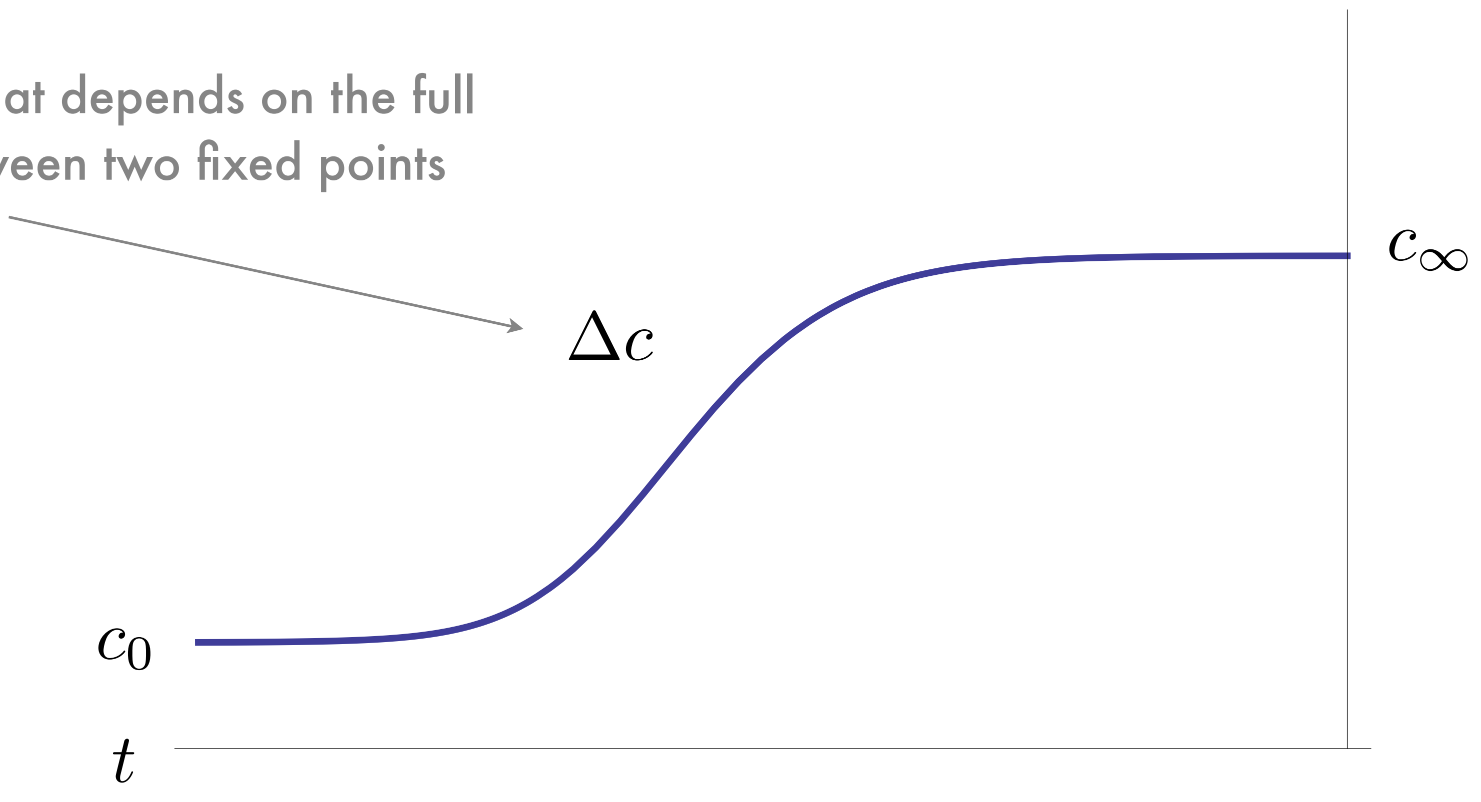
$$\partial_t c_k \geq 0$$



theory space

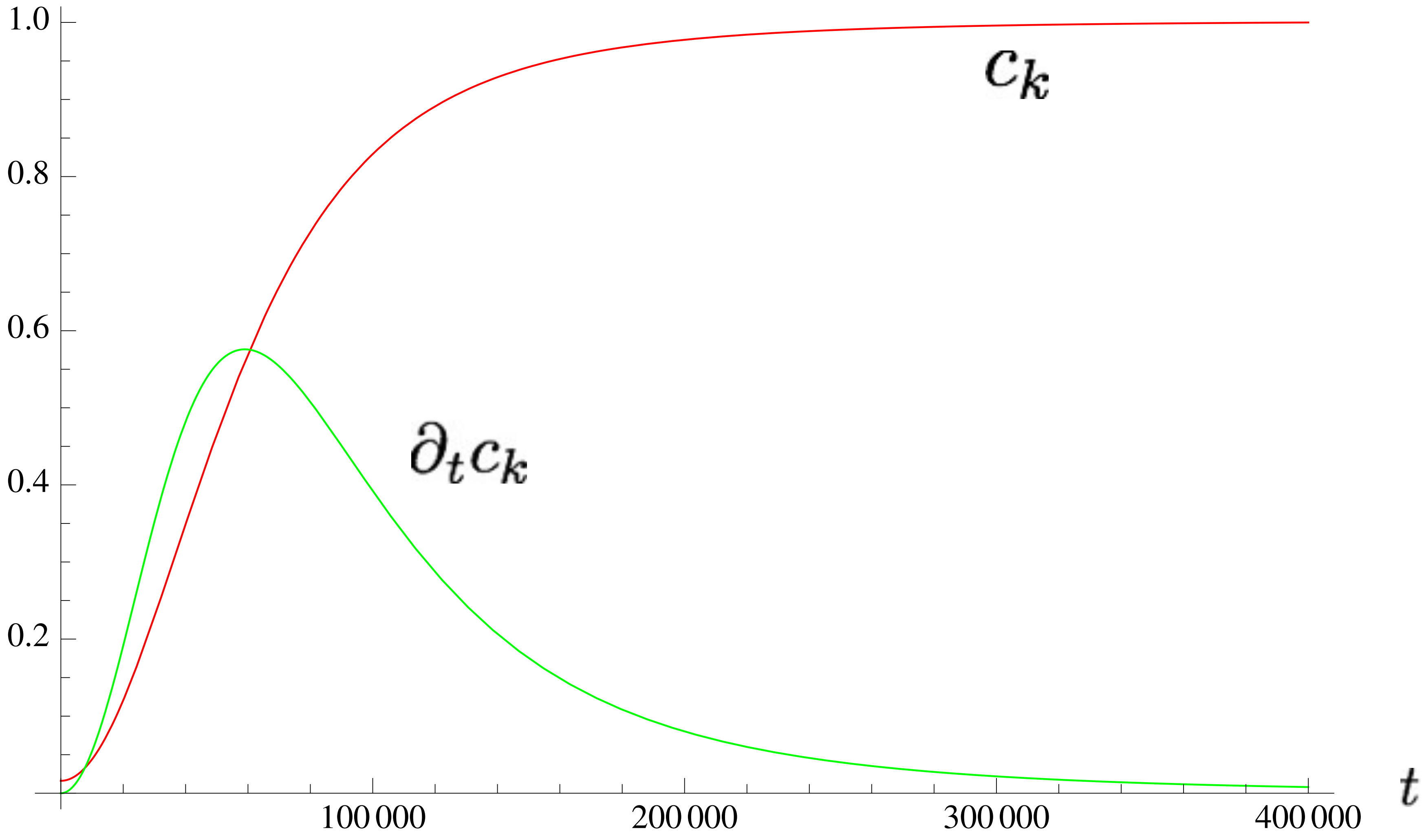

10
5
0
$\tilde{\lambda}_k$
0.4
0.2
0.0
$\partial_t c_k$
−0.5
0.0
0.5
$\tilde{m}_k^2$




universal quantity that depends on the full
RG trajectory between two fixed points

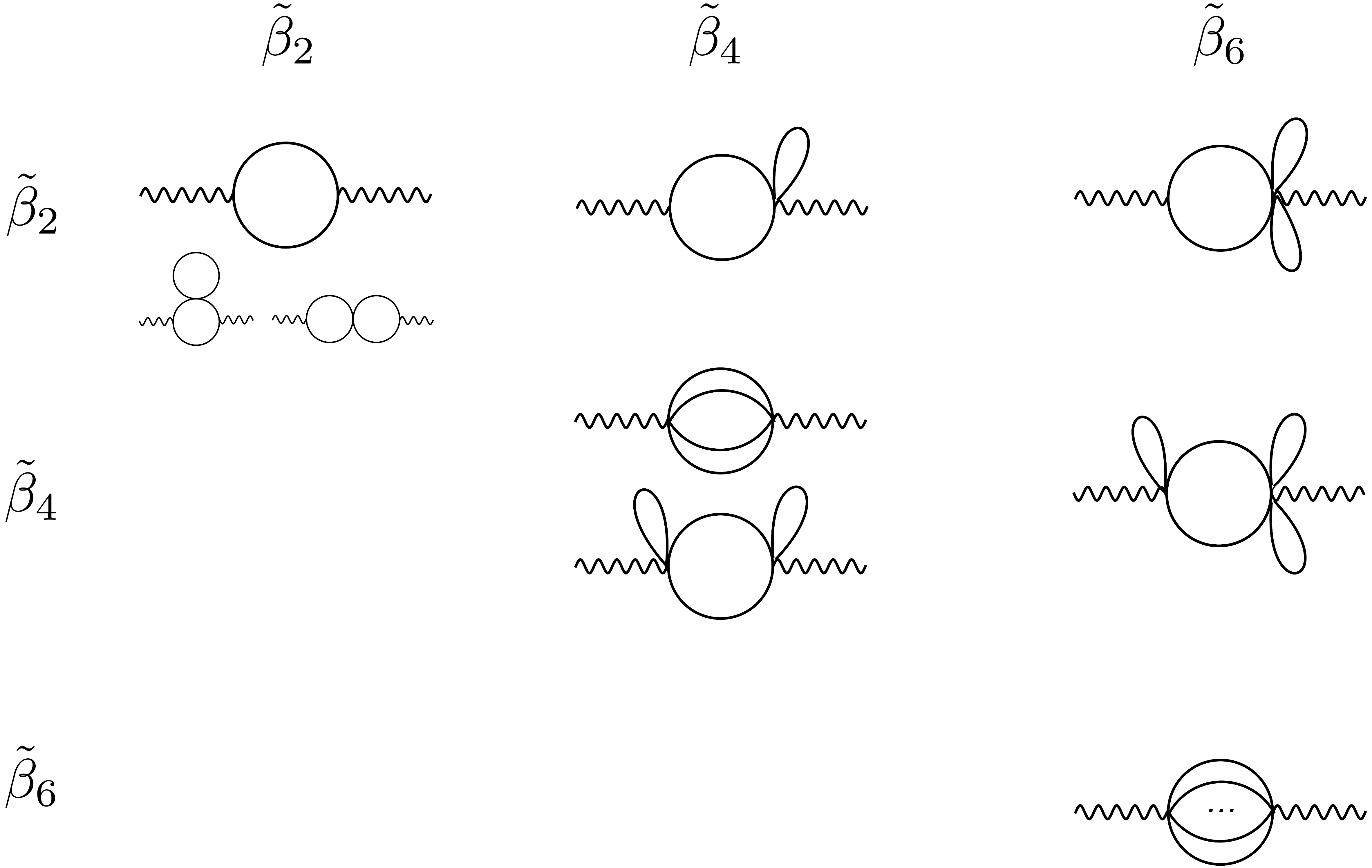
Sine-Gordon model

$$S_{SG}[\phi] = \int \left[\frac{1}{2}\phi\Delta\phi - \frac{m^2}{\beta^2}\left(\cos\left(\beta\phi\right) - 1\right)\right]$$



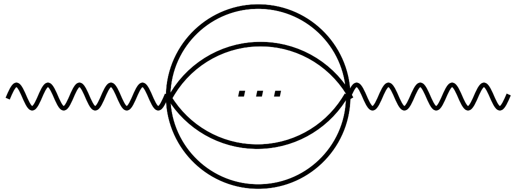
c- & a-functions in the loop expansion

c- & a-functions in the loop expansion



c- & a-functions in the loop expansion

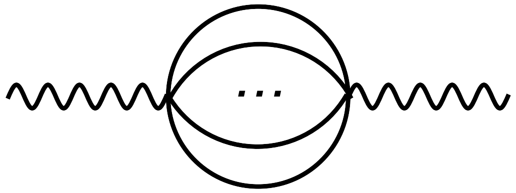
Diagonal contributions:



$$\partial_t \Gamma_{L,k} = -\frac{1}{2(L+1)!}\,\tilde{\beta}^2_{L+1}\,k^4 \int d^2x \int d^2y\,\tau_x \tau_y\,\tilde{\partial}_t\left[G_k\left(x-y\right)\right]^{L+1}$$

c- & a-functions in the loop expansion

Diagonal contributions:

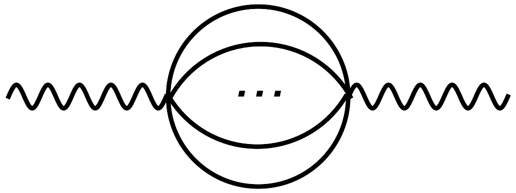


$$G_k\left(x-y\right)=\frac{1}{2\pi}K_0\left(|x-y|\sqrt{ak^2}\right)$$

$$\partial_t\Gamma_{L,k}=-\frac{1}{2(L+1)!}\,\tilde{\beta}_{L+1}^2\,k^4\int d^2x\int d^2y\,\tau_x\tau_y\,\tilde{\partial}_t\left[G_k\left(x-y\right)\right]^{L+1}$$

c- & a-functions in the loop expansion

Diagonal contributions:



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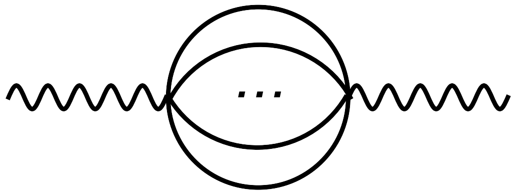
$$\downarrow$$

$$\partial_t\Gamma_{L,k}=\frac{k^4}{(L+1)!}\,\tilde{\beta}^2_{L+1}\int d^2x\,\tau_x\Delta\tau_x\int d^2y\frac{y^2}{2(2\pi)^{L+1}}\partial_a\left[K_0\left(|y|\sqrt{ak^2}\right)\right]^{L+1}\Bigg|_{a\to1}$$

c- & a-functions in the loop expansion

Diagonal contributions:



$$G_k\left(x-y\right) = \frac{1}{2\pi} K_0\left(|x-y|\sqrt{ak^2}\right)$$

$$\partial_t \Gamma_{L,k} = -\frac{1}{2(L+1)!}\,\tilde{\beta}^2_{L+1}\,k^4 \int d^2x \int d^2y\,\tau_x\tau_y\,\tilde{\partial}_t\left[G_k\left(x-y\right)\right]^{L+1}$$

$$\downarrow$$

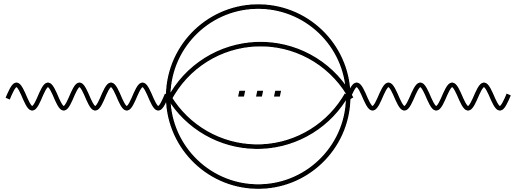
$$\partial_t \Gamma_{L,k} = \frac{k^4}{(L+1)!}\,\tilde{\beta}^2_{L+1} \int d^2x\,\tau_x\Delta\tau_x \int d^2y\,\frac{y^2}{2(2\pi)^{L+1}}\partial_a\left[K_0\left(|y|\sqrt{ak^2}\right)\right]^{L+1}\Bigg|_{a\to 1}$$

$$\downarrow$$

$$\partial_t c_{L,k} = \mathcal{A}_L\,\tilde{\beta}^2_{L+1}$$

c- & a-functions in the loop expansion

Diagonal contributions:



$$G_k\left(x-y\right)=\frac{1}{2\pi}K_0\left(|x-y|\sqrt{ak^2}\right)$$

$$\partial_t\Gamma_{L,k}=-\frac{1}{2(L+1)!}\,\tilde{\beta}^2_{L+1}\,k^4\int d^2x\int d^2y\,\tau_x\tau_y\,\tilde{\partial}_t\left[G_k\left(x-y\right)\right]^{L+1}$$

$$\downarrow$$

$$\partial_t\Gamma_{L,k}=\frac{k^4}{(L+1)!}\,\tilde{\beta}^2_{L+1}\int d^2x\,\tau_x\Delta\tau_x\int d^2y\,\frac{y^2}{2(2\pi)^{L+1}}\partial_a\left[K_0\left(|y|\sqrt{ak^2}\right)\right]^{L+1}\Bigg|_{a\to 1}$$

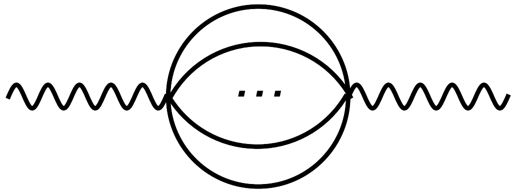
$$\downarrow$$

$$\partial_t c_{L,k}=\mathcal{A}_L\,\tilde{\beta}^2_{L+1}$$

$$\mathcal{A}_L\equiv\frac{3}{2^L\pi^{L-1}L!}\int_0^\infty dx\,x^4\left[K_0\left(x\right)\right]^L K_1\left(x\right)$$

c- & a-functions in the loop expansion

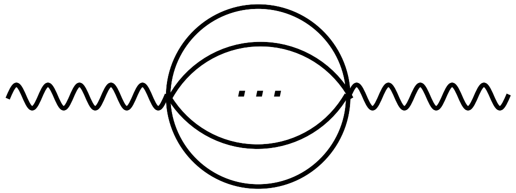
Diagonal contributions:



$$\mathcal{A}_L > 0$$

c- & a-functions in the loop expansion

Diagonal contributions:



$$\mathcal{A}_L > 0$$

$$\partial_t c_k^{(diagonal)} = \sum_{i=1}^{\infty} \mathcal{A}_{2i-1}\, \tilde{\beta}_{2i}^2$$

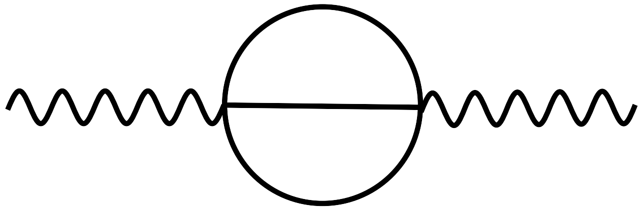
The c-theorem is satisfied by the diagonal contributions:

$$\partial_t c^{diagonal} > 0$$

c- & a-functions in the loop expansion

Non-unitary case:

$$S_{LY}[\phi] = \int d^2x \left[\frac{1}{2}\phi\Delta\phi + ig\phi^3\right]$$



$$\partial_t c_k = -\mathcal{A}_2\, \tilde{\beta}_3^2 < 0$$

$$\mathcal{A}_2 > 0$$

c- & a-functions in the loop expansion

d = 4

$$\partial_t a_k^{(diagonal)} = \mathcal{A}_3 \tilde{\beta}_4^2 + ...$$

$$\mathcal{A}_3 = \frac{1}{2^{12}\pi^6 \left(4!\right)^2}$$

Scheme independent!

$$\partial_t a_k^{diagonal} > 0$$

The a-theorem is valid in the loop expansion

Switch on gravity!

$$\mathcal{O} = R$$

$$\begin{aligned}\Gamma_k[g] = \int \sqrt{g}\,\Big[&-\frac{1}{16\pi G_k}R + ...\\ &-\frac{1}{4}\partial_t\left(-\frac{1}{16\pi G_k}\right)R\frac{1}{\Delta}R + ...\Big]\end{aligned}$$

$$\begin{aligned}= \int \sqrt{g}\,\Big[&-\frac{1}{16\pi G_k}R + ...\\ &-\frac{c_k - c_\Lambda}{96\pi}R\frac{1}{\Delta}R + ...\Big]\end{aligned}$$

Switch on gravity!

$$\mathcal{O} = R$$

$$\Gamma_k[g] = \int \sqrt{g} \left[-\frac{1}{16\pi G_k} R + \dots \right.$$

$$\left. -\frac{1}{4}\partial_t \left(-\frac{1}{16\pi G_k} \right) R \frac{1}{\Delta} R + \dots \right]$$

$$= \int \sqrt{g} \left[-\frac{1}{16\pi G_k} R + \dots \right.$$

$$\left. -\frac{c_k - c_\Lambda}{96\pi} R \frac{1}{\Delta} R + \dots \right]$$

$$\downarrow$$

$$\boxed{\partial_t \left(-\frac{1}{16\pi G_k} \right) = \frac{c_k - c_\Lambda}{24\pi}}$$

Switch on gravity!

$$\boxed{\partial_t c_k = \frac{3}{2G_k^2}\left(\partial_t \beta_{G_k} - 2\frac{\beta_{G_k}^2}{G_k}\right)}$$

Switch on gravity!

$$\boxed{\partial_t c_k = \frac{3}{2G_k^2}\left(\partial_t \beta_{G_k} - 2\frac{\beta_{G_k}^2}{G_k}\right)}$$

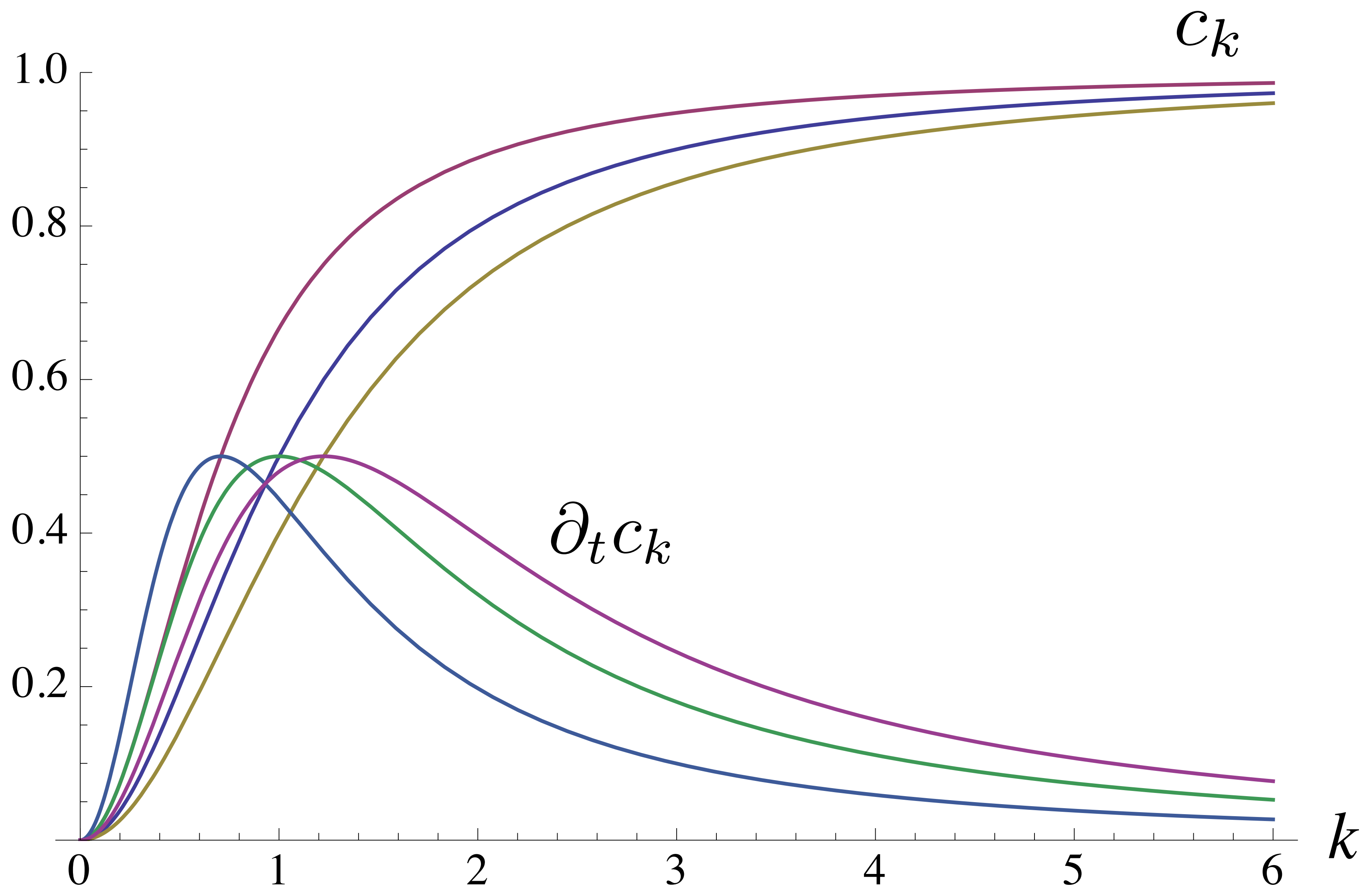
minimally coupled scalar:

$$c_k = \frac{ak^2}{ak^2 + bm^2} \qquad \partial_t c_k = \frac{2abk^2m^2}{\left(ak^2 + bm^2\right)^2}$$

$$R_k(z) = \frac{az}{e^{bz/k^2} - 1}$$

Switch on gravity!

minimally coupled scalar:



Switch on gravity!

interacting scalar:

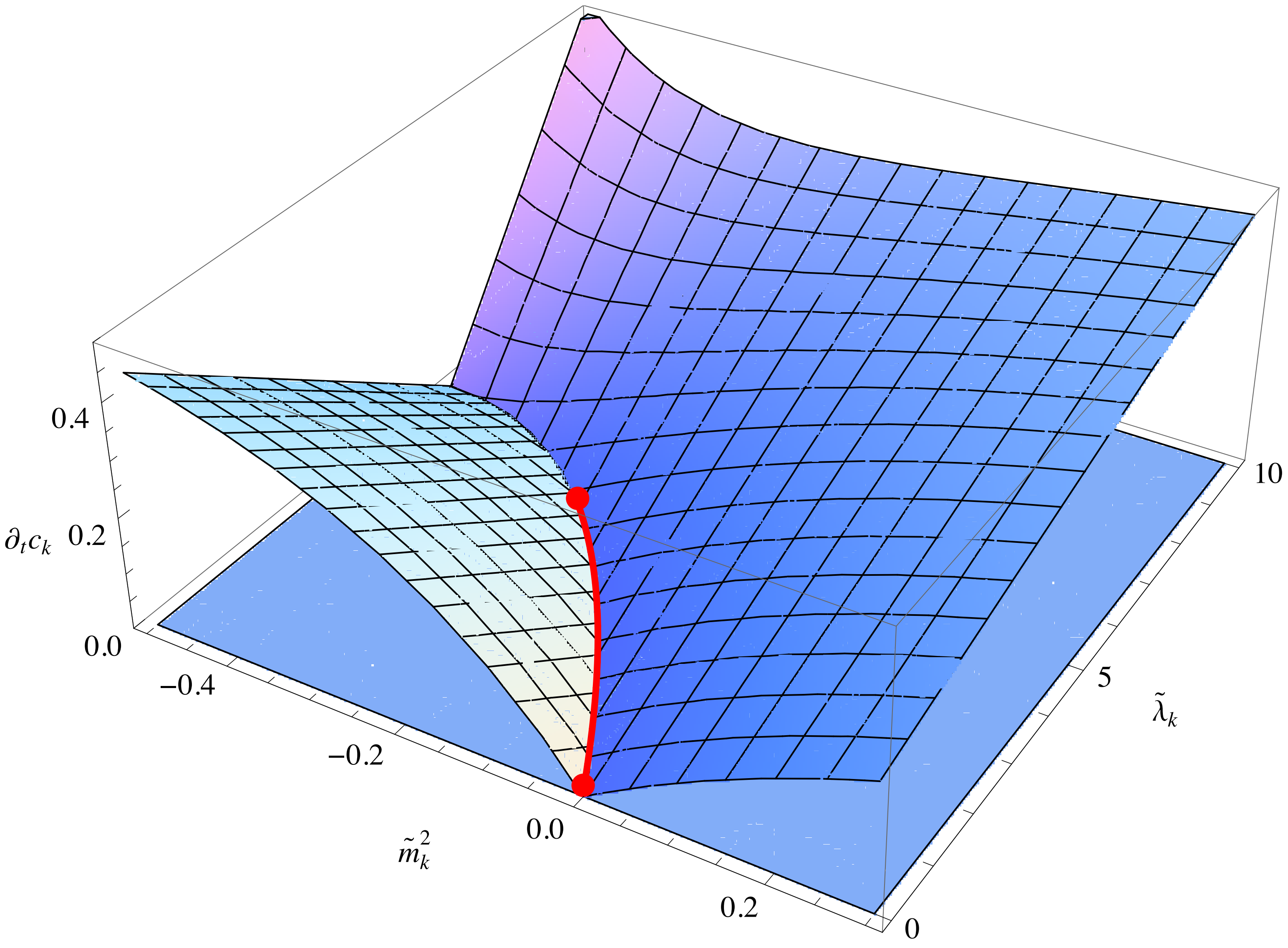
$$c_k = \frac{ak^2}{ak^2 + bV_k''(\varphi_0)}$$

$$\partial_t c_k = -\frac{abk^2\left(\partial_t V_k''(\varphi_0) - 2V_k''(\varphi_0)\right)}{\left(ak^2 + bV_k''(\varphi_0)\right)^2}$$

$$\partial_t c_k = \begin{cases} -\frac{ab\,\partial_t \tilde{m}_k^2}{(a+b\,\tilde{m}_k^2)^2} & \text{ordered phase} \\ \\ \frac{2ab\,\partial_t \tilde{m}_k^2}{(a-2b\,\tilde{m}_k^2)^2} & \text{broken phase}\,. \end{cases}$$

Switch on gravity!

interacting scalar:



Conclusions & Outlook

Understanding of how to parametrize
the effective (average) action
away from criticality

Non-perturbative definition of the c- and a-functions

Framework to calculate approximated c- and a-functions

A proof of the strong
c- and a-theorems using the fRG?

Thank you