GRENOBLE 1 UNIVERSITE JOSEPH FOURIER SCIENCES.TECHNOLOGIE.SANTÉ

cnrs dépasser les frontières

Fully developed isotropic turbulence from Navier-Stokes equations

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Presentation outline

1. Introduction : fully developed isotropic turbulence
2. Navier-Stokes equations, field theory and FRG formalism
3. Symmetries
4. Fixed point at Leading Order approximation
5. Exact flow equations in the large wave-number limit

Examples of fully developed turbulence

Ubiquitous phenomenon, experienced in everyday life

turbulence in the sea

turbulence in clouds

Examples of fully developed turbulence

turbulence in industrial smokes

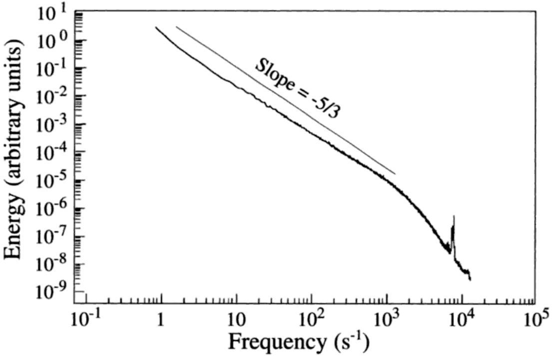
turbulence behind wind turbines

Scale invariance

longitudinal velocity structure functions

$$S_p(\ell) \equiv \langle (\delta v_\ell)^p \rangle \sim \ell^{\xi_p} \qquad \delta v_\ell = [\vec{u}(\vec{x}+\vec{\ell}) - \vec{u}(\vec{x})] \cdot \vec{\ell}$$

energy spectrum



ONERA
S1 wind tunnel

F. Anselmet, Y. Gagne,

E.J. Hopfinger,

R.A. Antonia,

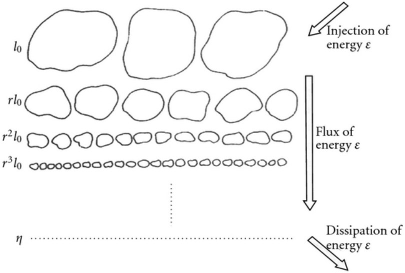
J. Fluid Mech. **140** (1984).

Kolmogorov K41 theory for isotropic 3D turbulence

assumptions

- homogeneity
- isotropy
- dimensional analysis
- self-similarity

A.N. Kolmogorov, Dokl. Akad. Nauk. SSSR **30, 31, 32** (1941)
U. Frisch, *Turbulence*, Cambridge Univ. Press (1995)
M. Lesieur, *Turbulence in fluids*, Springer



predictions

$$S_p(\ell) = C_p\, \epsilon^{p/3}\, \ell^{p/3}$$

$$S_3(\ell) = -\frac{4}{5}\, \epsilon\, \ell$$

$$E(k) = C_K\, \epsilon^{2/3}\, k^{-5/3}$$

Intermittency

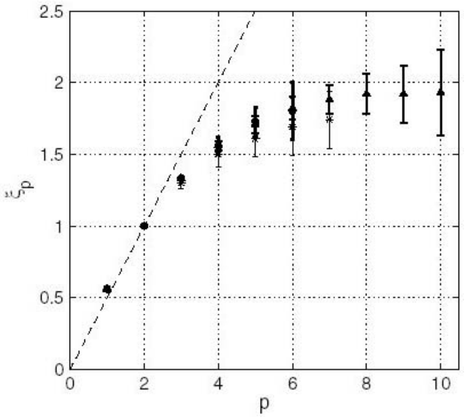
deviations from K41

from experiments and numerical simulations

$$\xi_p \neq p/3$$

U. Frisch, *Turbulence*, Cambridge Univ. Press (1995)
M. Lesieur, *Turbulence in fluids*, Springer (2008)

illustration : *von Kárman swirling flow*



N. Mordant, E. Lévêque, J.-F. Pinton, New J. Phys. **6** (2004)

● experiment
▲ , * DNS
- - - K41

Aims of the presentation

Feynman's words about turbulence as
"the most important unsolved problem of classical physics"
still valid

$\longrightarrow$ full understanding of turbulence *from first principles*
still lacking despite huge literature and many results

textbooks on turbulence, (RG) Zhou, Phys. Rep. **488** (2010), (FRG) talk of Steven Mathey

our contribution

- re-visit FRG formalism
- bring out importance of symmetries
- careful analysis of the fixed point in $d = 2$ and $d = 3$
- exact flow equations for the 2-point correlation functions *in the large momentum regime*

L. Canet, B. Delamotte, N. Wschebor, *to appear*

Microscopic theory

Navier Stokes equation with forcing for incompressible fluids

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \nu \nabla^2 \vec{v} + \vec{f}$$

$$\vec{\nabla} \cdot \vec{v}(t, \vec{x}) = 0$$

- $\vec{v}(\vec{x}, t)$ velocity field and $p(\vec{x}, t)$ pressure field
- ρ density and ν kinematic viscosity
- $\vec{f}(\vec{x}, t)$ gaussian stochastic stirring force with variance

$$\left\langle f_\alpha(t, \vec{x}) f_\beta(t', \vec{x}\,') \right\rangle = 2\delta(t - t') N_{L^{-1},\alpha\beta}(|\vec{x} - \vec{x}\,'|).$$

with L the integral scale (energy injection)

Field theory

Janssen de Dominicis formalism

introduction of Martin-Siggia-Rose response fields $\bar{\vec{v}}$ and $\bar{p}$ to integrate out the stochastic forcing

$$\mathcal{Z}[\vec{J},\bar{\vec{J}},K,\bar{K}] = \int \mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{D}\bar{\vec{v}}\,\mathcal{D}\bar{p}\; e^{-(\mathcal{S}_0[\vec{v},\bar{\vec{v}},p,\bar{p}]+\Delta\mathcal{S}_0[\vec{v},\bar{\vec{v}}])}$$

$$\times\; e^{\int_{t,\vec{x}}\{\vec{J}\cdot\vec{v}+\bar{\vec{J}}\cdot\bar{\vec{v}}+K\,p+\bar{K}\,\bar{p}\}} \quad \longleftarrow \text{sources}$$

deterministic NS equation ↴

$$\mathcal{S}_0[\vec{v},\bar{\vec{v}},p,\bar{p}] = \int_{t,\vec{x}} \bar{v}_\alpha \times \boxed{\partial_t v_\alpha + v_\beta\partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha p - \nu\nabla^2 v_\alpha}$$

$$+\int_{t,\vec{x}} \bar{p} \times \boxed{\partial_\alpha v_\alpha} \quad \longleftarrow \text{incompressibility constraint}$$

$$\Delta\mathcal{S}_0[\vec{v},\bar{\vec{v}}] = -\int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha \boxed{N_{L^{-1},\alpha\beta}(|\vec{x}-\vec{x}'|)}\, \bar{v}_\beta \quad \longleftarrow \text{force correlator}$$

FRG formalism I

regulator term

$$\Delta\mathcal{S}_\kappa[\vec{v},\bar{\vec{v}}] = -\int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x}) N_{\kappa,\alpha\beta}(|\vec{x}-\vec{x}'|)\bar{v}_\beta(t,\vec{x}')$$

R. Collina and P. Tomassini, Phys. Lett. B **411** (1997)

$$N_{\kappa,\alpha\beta}(\vec{q}) = \delta_{\alpha\beta}\, D_\kappa\, (|\vec{q}|/\kappa)^2\, \hat{n}\,(|\vec{q}|/\kappa)$$

$$\hat{n}(x) = e^{-x^2}$$

to maintain a stationary turbulent flow : $D_\kappa \sim \kappa^{-d}$

not enough to regulate the flow in $d = 2$

FRG formalism I

regulator term

$$\Delta\mathcal{S}_\kappa[\vec{v},\bar{\vec{v}}] = -\int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x}) N_{\kappa,\alpha\beta}(|\vec{x}-\vec{x}'|)\bar{v}_\beta(t,\vec{x}') + \int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x}) R_{\kappa,\alpha\beta}(|\vec{x}-\vec{x}'|) v_\beta(t,\vec{x}')$$

L. Canet, B. Delamotte and N. Wschebor, *to appear*

$$R_{\kappa,\alpha\beta}(\vec{q}) = \delta_{\alpha\beta}\,\nu_\kappa\,\vec{q}^{\,2}\,\hat{r}\left(|\vec{q}|/\kappa\right)$$

$$\hat{r}(x) = a/(e^{x^2}-1)$$

scaling is fixed : $\nu_\kappa \sim \kappa^{-4/3}$

flow regulated in $d=2$ and possible independent scales

FRG formalism II

Legendre transform and effective action

$$\Gamma_\kappa[\vec u,\vec{\bar u},p,\bar p]+\mathcal{W}_\kappa[\vec J,\vec{\bar J},K,\bar K]=\int_{t,\vec x}\vec u\cdot\vec J+\vec{\bar u}\cdot\vec{\bar J}+p\,K+\bar p\,\bar K$$
$$-\int_{t,\vec x,\vec x'}\Big\{\bar u_\alpha\,R_{\kappa,\alpha\beta}\,u_\beta-\bar u_\alpha\,N_{\kappa,\alpha\beta}\,\bar u_\beta\Big\}$$

$$u_\alpha\equiv\langle v_\alpha\rangle=\frac{\delta\mathcal{W}_\kappa}{\delta J_\alpha}\qquad\qquad \bar u_\alpha\equiv\langle\bar v_\alpha\rangle=\frac{\delta\mathcal{W}_\kappa}{\delta\bar J_\alpha}$$

Wetterich's equation for the 2-point functions

$$\partial_\kappa\Gamma^{(2)}_{\kappa,ij}(\mathbf p)\;=\;\mathrm{Tr}\int_{\mathbf q}\partial_\kappa\mathcal{R}_\kappa(\mathbf q)\cdot G_\kappa(\mathbf q)\cdot\Big(-\frac12\,\Gamma^{(4)}_{\kappa,ij}(\mathbf p,-\mathbf p,\mathbf q)$$
$$+\Gamma^{(3)}_{\kappa,i}(\mathbf p,\mathbf q)\cdot G_\kappa(\mathbf p+\mathbf q)\cdot\Gamma^{(3)}_{\kappa,j}(-\mathbf p,\mathbf p+\mathbf q)\Big)\cdot G_\kappa(\mathbf q)$$

Symmetry I

pressure sector of the NS field theory

$$\int_{t,\vec{x}} \left\{ \frac{1}{\rho}\, \bar{v}_\alpha\, \partial_\alpha p + \bar{p}\, \partial_\alpha v_\alpha + K\, p + \bar{K}\, \bar{p} \right\}$$

infinitesimal gauged shifts of the pressure and response pressure

$$p(t,\vec{x}) \to p(t,\vec{x}) + \epsilon(t,\vec{x})$$
$$\bar{p}(t,\vec{x}) \to \bar{p}(t,\vec{x}) + \bar{\epsilon}(t,\vec{x})$$

variation is linear in the fields $\longrightarrow$ Ward identities

$$\frac{\delta\Gamma_\kappa}{\delta p(t,\vec{x})} = \frac{\delta S_0}{\delta p(t,\vec{x})} \qquad \text{and} \qquad \frac{\delta\Gamma_\kappa}{\delta \bar{p}(t,\vec{x})} = \frac{\delta S_0}{\delta \bar{p}(t,\vec{x})}$$

Non-renormalisation of the pressure sector

Symmetry II

infinitesimal time-gauged galilean transformations

$$\mathcal{G}(\vec{\epsilon}(t)) = \begin{cases} \delta v_\alpha(t,\vec{x}) & = -\dot{\epsilon}_\alpha(t) + \epsilon_\beta(t)\partial_\beta v_\alpha(t,\vec{x}) \\ \delta \bar{v}_\alpha(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta \bar{v}_\alpha(t,\vec{x}) \\ \delta p(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta p(t,\vec{x}) \\ \delta \bar{p}(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta \bar{p}(t,\vec{x}) \end{cases}$$

$\mathcal{G}(\vec{\epsilon}) =$ translation $\qquad \mathcal{G}(\vec{\epsilon}\,t) =$ galilean transformation

NS action is invariant under $\mathcal{G}(\vec{\epsilon}(t))$ but for

$$\delta\mathcal{S} = \delta\Big\{\int_{t,\vec{x}} \bar{v}_\alpha D_t v_\alpha\Big\} = -\int_{t,\vec{x}} \ddot{\epsilon}_\alpha(t)\bar{v}_\alpha$$

$D_t v_\alpha \equiv \partial_t v_\alpha + v_\beta \partial_\beta v_\alpha$ *Lagrangian time derivative*

Non-renormalisation of $\bar{v}_\alpha D_t v_\alpha$ and invariance under $\mathcal{G}(\vec{\epsilon}(t))$ of the rest of the effective action

Symmetry III

infinitesimal time-gauged response field shift

$$\mathcal{R}(\vec{\bar{\epsilon}}(t)) = \begin{cases} \delta\bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta\bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t) \end{cases}$$

variation of the NS action (at most) linear in the fields
$\longrightarrow$ Ward identities

Non-renormalisation of $\bar{v}_\alpha \partial_t v_\alpha$ and invariance under $\mathcal{R}(\vec{\bar{\epsilon}}(t))$ of the rest of the effective action

Symmetry : summary

general form of the effective action

$$\Gamma_\kappa[\vec{u},\bar{\vec{u}},p,\bar{p}] = \int_{t,\vec{x}} \Big\{ \bar{u}_\alpha \Big(\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha + \frac{\partial_\alpha p}{\rho} \Big) + \bar{p}\, \partial_\alpha u_\alpha \Big\} + \hat{\Gamma}_\kappa[\vec{u},\bar{\vec{u}}]$$

with $\hat{\Gamma}_\kappa[\vec{u},\bar{\vec{u}}]$ invariant under the two gauged symmetries

we know how to construct it from experience on KPZ equation !

- similar nonlinear Langevin equation (equivalent to Burgers)
- very similar (gauged) symmetries

LO approximation very successful (*talk of Thomas Kloss*)
$\longrightarrow$ quadratic in the fields with full momentum dependence

L. Canet, H. Chaté, B. Delamotte, N. Wschebor, Phys. Rev. Lett. **104** (2010), Phys. Rev. E **84** (2011)
T. Kloss, L. Canet, N. Wschebor, Phys. Rev. E **86** (2012), TK, LC, BD, NW, Phys. Rev. E, **89** (2014)

Composite operator

source for the composite operator $v_\alpha(\vec{x},t)v_\beta(\vec{x},t)$

$$\mathcal{Z}[\vec{J},\bar{\vec{J}},K,\bar{K},L] \propto e^{\int_{t,\vec{x}}\{\vec{J}\cdot\vec{v}+\bar{\vec{J}}\cdot\bar{\vec{v}}+Kp+\bar{K}\bar{p}+\vec{v}\cdot L\cdot\vec{v}\}}$$

infinitesimal response field shift gauged in time *and space*

$$\mathcal{R}(\vec{\bar{\epsilon}}(t,\vec{x})) = \begin{cases} \delta\bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t,\vec{x}) \\ \delta\bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t,\vec{x}) \end{cases}$$

local Ward identity for Γ_κ

$$\frac{\delta\Gamma_\kappa}{\delta\bar{u}_\alpha} = \partial_t u_\alpha + \frac{1}{\rho}\partial_\alpha p - \nu\nabla^2 u_\alpha - \partial_\beta\left(\frac{\delta\Gamma_\kappa}{\delta L_{\alpha\beta}}\right) - u_\alpha\partial_\beta u_\beta$$

generalized response function *vertex functions*

source for the composite operator $v_\alpha(\vec{x},t)v_\beta(\vec{x},t)$

$$\mathcal{Z}[\vec{J},\vec{\bar{J}},K,\bar{K},L] \quad \propto \quad e^{\int_{t,\vec{x}}\{\vec{J}\cdot\vec{v}+\vec{\bar{J}}\cdot\vec{\bar{v}}+Kp+\bar{K}\bar{p}+\vec{v}\cdot L\cdot\vec{v}\}}$$

infinitesimal response field shift gauged in time *and space*

$$\mathcal{R}(\vec{\bar{\epsilon}}(t,\vec{x})) = \begin{cases} \delta\bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t,\vec{x}) \\ \delta\bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t,\vec{x}) \end{cases}$$

local Ward identity for $\mathcal{W}_\kappa$

$$\big[-\partial_t+\nu\nabla^2+\bar{K}\big]\frac{\delta\mathcal{W}_\kappa}{\delta J_\alpha}-\frac{1}{\rho}\partial_\alpha\frac{\delta\mathcal{W}_\kappa}{\delta K}\bar{J}_\alpha-\partial_\beta\frac{\delta\mathcal{W}_\kappa}{\delta L_{\alpha\beta}}+\int_{\vec{x}'}\Big\{2\frac{\delta\mathcal{W}_\kappa}{\delta\bar{J}_\beta}N_{\kappa,\alpha\beta}+\frac{\delta\mathcal{W}_\kappa}{\delta J_\beta}R_{\kappa,\alpha\beta}\Big\}=0$$

- derivative w.r.t. $J_\beta \Longrightarrow$ Kárman-Howarth-Monin relation

$\Longrightarrow$ four-fifth Kolmogorov law : $S_3(\ell) = -\frac{4}{5}\,\epsilon\,\ell$

Composite operator

source for the composite operator $v_\alpha(\vec{x},t)v_\beta(\vec{x},t)$

$$\mathcal{Z}[\vec{J},\vec{\bar{J}},K,\bar{K},L] \quad \propto \quad e^{\int_{t,\vec{x}}\{\vec{J}\cdot\vec{v}+\vec{\bar{J}}\cdot\vec{\bar{v}}+Kp+\bar{K}\bar{p}+\vec{v}\cdot L\cdot\vec{v}\}}$$

infinitesimal response field shift gauged in time *and* space

$$\mathcal{R}(\vec{\bar{\epsilon}}(t,\vec{x})) = \begin{cases} \delta\bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t,\vec{x}) \\ \delta\bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t,\vec{x}) \end{cases}$$

***local* Ward identity for $\mathcal{W}_\kappa$**

$$\big[-\partial_t+\nu\nabla^2+\bar{K}\big]\frac{\delta\mathcal{W}_\kappa}{\delta J_\alpha}-\frac{1}{\rho}\partial_\alpha\frac{\delta\mathcal{W}_\kappa}{\delta K}\bar{J}_\alpha-\partial_\beta\frac{\delta\mathcal{W}_\kappa}{\delta L_{\alpha\beta}}+\int_{\vec{x}'}\Big\{2\frac{\delta\mathcal{W}_\kappa}{\delta\bar{J}_\beta}N_{\kappa,\alpha\beta}+\frac{\delta\mathcal{W}_\kappa}{\delta J_\beta}R_{\kappa,\alpha\beta}\Big\}=0$$

- derivatives w.r.t. arbitrary sources

$\Longrightarrow$ infinite set of generalized exact relations

Ansatz for $\hat\Gamma_\kappa$

ansatz invariant under the symmetries at Leading Order

$$\hat\Gamma_\kappa[\vec u, \bar{\vec u}] = \int_{t,\vec x,\vec x'} \Big\{ \bar u_\alpha\, f^\nu_{\kappa,\alpha\beta}(\vec x - \vec x')\, u_\beta - \bar u_\alpha\, f^D_{\kappa,\alpha\beta}(\vec x - \vec x')\, \bar u_\beta \Big\}$$

$$\text{with } f^\nu_{\alpha\beta}(\vec p = \vec 0) = f^D_{\alpha\beta}(\vec p = \vec 0) = 0$$

momentum dependent two-point functions

$$\hat\Gamma^{(2,0)}_{\alpha\beta}(\omega, \vec p) = 0$$

$$\hat\Gamma^{(1,1)}_{\alpha\beta}(\omega, \vec p) = i\omega\delta_{\alpha\beta} + f^\nu_{\alpha\beta}(\vec p),$$

$$\hat\Gamma^{(0,2)}_{\alpha\beta}(\omega, \vec p) = -2\, f^D_{\alpha\beta}(\vec p)$$

one non-vanishing vertex function

$$\Gamma^{(2,1)}_{\alpha\beta\gamma}(\omega_1, \vec p_1, \omega_2, \vec p_2) = -i(p_2^\alpha \delta_{\beta\gamma} + p_1^\beta \delta_{\alpha\gamma})$$

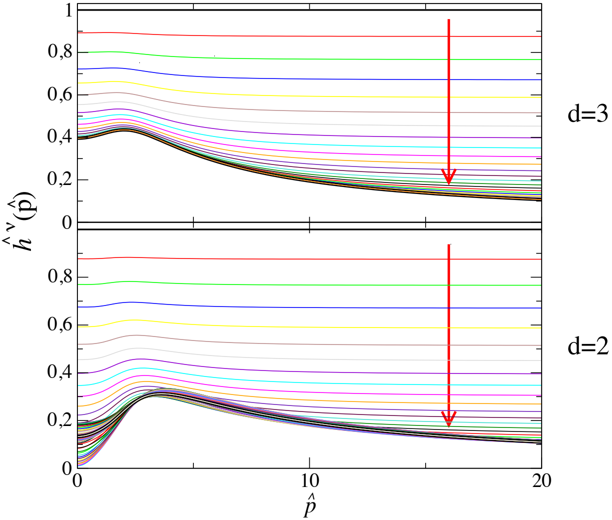
Numerical integration of the LO flow equations

RG evolution

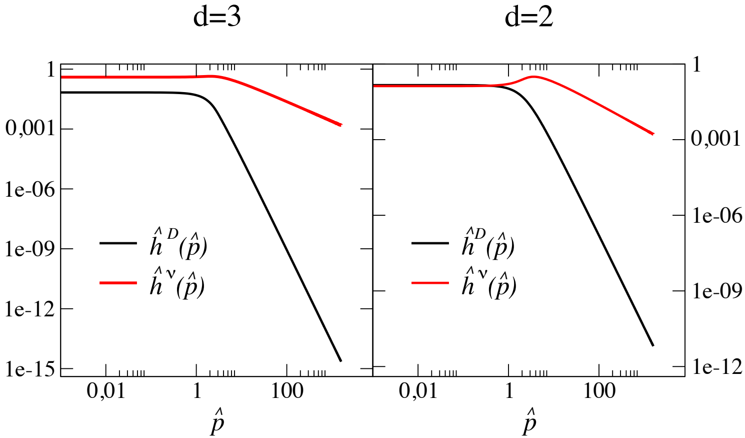
dimensionless functions

$\hat{h}^{\nu}(\hat{p}) \equiv \hat{f}_{\perp}^{\nu}(\hat{p})/\hat{p}^2$

fixed point



Fixed point functions

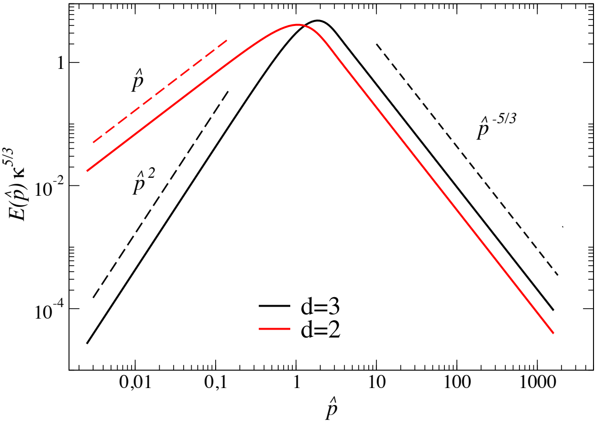


asymptotics in $d = 3$: deviations from naive scaling

$$\hat{h}^\nu(\hat{p}) \sim \hat{p}^{-4/3+\alpha} \qquad \alpha \simeq 0.33$$

$$\hat{h}^D(\hat{p}) \sim \hat{p}^{-(d+2)+\beta} \qquad \beta \simeq 0.33$$

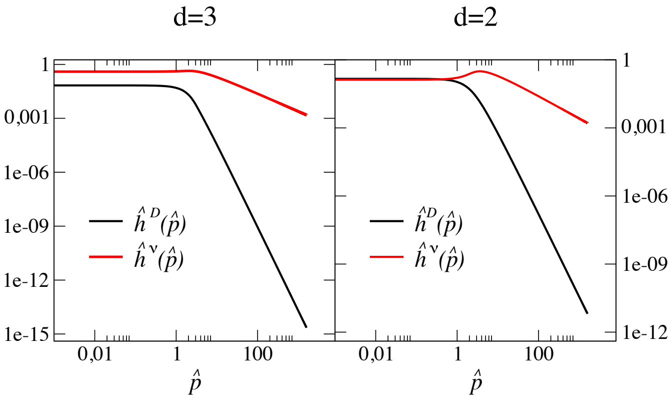
Energy spectrum and second order structure function



Kolmogorov scaling

energy spectrum	$E(\hat{p})$	$\sim$	$p^{-5/3}(p/\kappa)^{\beta-\alpha}$
structure function	$S_2(\ell)$	$\sim$	$\ell^{2/3}(\kappa\ell)^{\alpha-\beta}$

Analysis of the large wave-number regime



α and β *universal* (independent of the stirring profile)
cf. R. Collina and P. Tomassini
$\Longrightarrow$ non-decoupling of the large momentum sector
but Leading Order approximation *not reliable* in this regime !

Analysis of the large wave-number regime

Wetterich's equation for the 2-point functions

$$\partial_\kappa \Gamma^{(2)}_{\kappa,ij}(\mathbf{p}) = \mathrm{Tr}\int_{\mathbf{q}} \partial_\kappa \mathcal{R}_\kappa(\mathbf{q}) \cdot G_\kappa(\mathbf{q}) \cdot \Big(-\frac{1}{2}\Gamma^{(4)}_{\kappa,ij}(\mathbf{p},-\mathbf{p},\mathbf{q})$$
$$+\Gamma^{(3)}_{\kappa,i}(\mathbf{p},\mathbf{q}) \cdot G_\kappa(\mathbf{p}+\mathbf{q}) \cdot \Gamma^{(3)}_{\kappa,j}(-\mathbf{p},\mathbf{p}+\mathbf{q}) \Big) \cdot G_\kappa(\mathbf{q})$$

LO approximation

$\longrightarrow$ expansion of the vertices in momentum

- *internal* momentum cut off $|\vec{q}| \lesssim \kappa$
- but controlled only for small *external* momentum $|\vec{p}| \lesssim \kappa$

Exact flow equations in the large wave-number limit I

Wetterich's equation for the 2-point functions

$$\partial_\kappa \Gamma^{(2)}_{\kappa,ij}(\mathbf{p}) = \mathrm{Tr}\int_{\mathbf{q}} \partial_\kappa \mathcal{R}_\kappa(\mathbf{q})\cdot G_\kappa(\mathbf{q})\cdot\Big(-\frac{1}{2}\Gamma^{(4)}_{\kappa,ij}(\mathbf{p},-\mathbf{p},\mathbf{q}) + \Gamma^{(3)}_{\kappa,i}(\mathbf{p},\mathbf{q})\cdot G_\kappa(\mathbf{p}+\mathbf{q})\cdot\Gamma^{(3)}_{\kappa,j}(-\mathbf{p},\mathbf{p}+\mathbf{q})\Big)\cdot G_\kappa(\mathbf{q})$$

regime of large wave-vector $|\vec{p}| \gg \kappa$ or $\kappa \to 0$

$\Longrightarrow$ internal momentum negligible $|\vec{q}| \ll |\vec{p}|$

exact Ward identities for all vertices with one zero momentum

$$\Gamma^{(2,1)}_{\alpha\beta\gamma}(\omega,\vec{q}=\vec{0};\nu,\vec{p}) = -\frac{p^\alpha}{\omega}\Big(\Gamma^{(1,1)}_{\beta\gamma}(\omega+\nu,\vec{p}) - \Gamma^{(1,1)}_{\beta\gamma}(\nu,\vec{p})\Big)$$

$$\Gamma^{(2,2)}_{\alpha\beta\gamma\delta}(\omega,\vec{0},-\omega,\vec{0},\nu,\vec{p}) = \frac{p^\alpha p^\beta}{\omega^2}\Big[\Gamma^{(0,2)}_{\gamma\delta}(\nu+\omega,\vec{p}) - 2\Gamma^{(0,2)}_{\gamma\delta}(\nu,\vec{p}) + \Gamma^{(0,2)}_{\gamma\delta}(\nu-\omega,\vec{p})\Big]$$

Exact flow equations in the large wave-number limit II

flow equations for the two-point functions

$$\partial_s \Gamma_\perp^{(1,1)}(\nu,\vec p) = p^2 \int_\omega \Bigg\{ -\left[\frac{\Gamma_\perp^{(1,1)}(\omega+\nu,\vec p) - \Gamma_\perp^{(1,1)}(\nu,\vec p)}{\omega}\right]^2 G_\perp^{u\bar u}(-\omega-\nu,\vec p)$$

$$+ \frac{1}{2\omega^2}\left[\Gamma_\perp^{(1,1)}(\omega+\nu,\vec p) - 2\Gamma_\perp^{(1,1)}(\nu,\vec p) + \Gamma_\perp^{(1,1)}(-\omega+\nu,\vec p)\right]\Bigg\}$$

$$\times \frac{(d-1)}{d}\tilde\partial_s \int_{\vec q} G_\perp^{uu}(\omega,\vec q)$$

$$\partial_s \Gamma_\perp^{(0,2)}(\nu,\vec p) = \dots$$

exact closed equations for large $\vec p$

Non-decoupling of the large wave-number sector

flow equation in terms of dimensionless quantities

$$\partial_s \Gamma^{(1,1)}(\nu, p) = \kappa^2 \nu_\kappa \Big\{ \partial_s \hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p})$$

$$+ \tfrac{2}{3} \hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \hat{p}\partial_{\hat{p}} \hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \tfrac{2}{3} \hat{\nu}\partial_{\hat{\nu}} \hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) \Big\}$$

Non-decoupling of the large wave-number sector

flow equation in terms of dimensionless quantities

$$\partial_s \Gamma^{(1,1)}(\nu,p) \; = \kappa^2 \nu_\kappa \Big\{ \boxed{\partial_s \hat\Gamma^{(1,1)}(\hat\nu,\hat p)}$$

$$+ \; \tfrac{2}{3}\hat\Gamma^{(1,1)}(\hat\nu,\hat p) - \hat p \partial_{\hat p}\hat\Gamma^{(1,1)}(\hat\nu,\hat p) - \tfrac{2}{3}\hat\nu\partial_{\hat\nu}\hat\Gamma^{(1,1)}(\hat\nu,\hat p) \; \Big\}$$

fixed point $\quad \boxed{\partial_s \hat\Gamma^{(1,1)}} \longrightarrow 0$

Non-decoupling of the large wave-number sector

flow equation in terms of dimensionless quantities

$$\boxed{\partial_s \Gamma^{(1,1)}(\nu, p)} = \kappa^2 \nu_\kappa \Big\{ \partial_s \hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p})$$
$$+ \frac{2}{3}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \hat{p}\partial_{\hat{p}}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \frac{2}{3}\hat{\nu}\partial_{\hat{\nu}}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) \Big\}$$

fixed point $\boxed{\partial_s \hat{\Gamma}^{(1,1)}} \longrightarrow 0$

decoupling $\boxed{\partial_s \Gamma^{(1,1)}} \longrightarrow 0$

Non-decoupling of the large wave-number sector

flow equation in terms of dimensionless quantities

$$\partial_s \Gamma^{(1,1)}(\nu,p) = \kappa^2 \nu_\kappa \Big\{ \partial_s \hat\Gamma^{(1,1)}(\hat\nu,\hat p) + \boxed{\tfrac{2}{3}\hat\Gamma^{(1,1)}(\hat\nu,\hat p) - \hat p \partial_{\hat p}\hat\Gamma^{(1,1)}(\hat\nu,\hat p) - \tfrac{2}{3}\hat\nu\partial_{\hat\nu}\hat\Gamma^{(1,1)}(\hat\nu,\hat p)} \Big\}$$

fixed point $\quad \boxed{\partial_s \hat\Gamma^{(1,1)}} \longrightarrow 0$

$+$

decoupling $\quad \boxed{\partial_s \Gamma^{(1,1)}} \longrightarrow 0$

$=$

$$\boxed{\tfrac{2}{3}\hat\Gamma^{(1,1)} - \hat p \partial_{\hat p}\hat\Gamma^{(1,1)} - \tfrac{2}{3}\hat\nu\partial_{\hat\nu}\hat\Gamma^{(1,1)}} \longrightarrow 0$$

Non-decoupling of the large wave-number sector

flow equation in terms of dimensionless quantities

$$\partial_s \Gamma^{(1,1)}(\nu, p) = \kappa^2 \nu_\kappa \Big\{ \partial_s \hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p})$$

$$+ \tfrac{2}{3}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \hat{p}\partial_{\hat{p}}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \tfrac{2}{3}\hat{\nu}\partial_{\hat{\nu}}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) \Big\}$$

fixed point $\boxed{\partial_s \hat{\Gamma}^{(1,1)}} \longrightarrow 0$

+

decoupling $\boxed{\partial_s \Gamma^{(1,1)}} \longrightarrow 0$

=

scale invariance $\boxed{\Gamma^{(1,1)}(\nu, p) = p^{2/3}\chi^{(1,1)}\left(\nu/p^{2/3}\right)}$

Non-decoupling of the large wave-number sector

flow equation in terms of dimensionless quantities

$$\partial_s \Gamma^{(1,1)}(\nu, p) = \kappa^2 \nu_\kappa \Big\{ \partial_s \hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) + \tfrac{2}{3}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \hat{p}\partial_{\hat{p}}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) - \tfrac{2}{3}\hat{\nu}\partial_{\hat{\nu}}\hat{\Gamma}^{(1,1)}(\hat{\nu}, \hat{p}) \Big\}$$

fixed point $\boxed{\partial_s \hat{\Gamma}^{(1,1)}} \longrightarrow 0$

$+$

decoupling $\boxed{\partial_s \Gamma^{(1,1)}} \longrightarrow 0$

$=$

scale invariance $\boxed{\Gamma^{(1,1)}(\nu, p) = p^{2/3}\chi^{(1,1)}\left(\nu/p^{2/3}\right)}$

but not consistent in the exact equation

$\Longrightarrow$ the large $\vec{p}$ sector does not decouple

Origin of intermittency

non-decoupling

- very particular ($\neq$ critical phenomena)
- probably general for all n-point functions
- correlation functions remain sensitive to the integral scale and may each have their own scaling

intermittency

- fixed point
  $\Longrightarrow$ power-law behaviour of the correlation functions
- no decoupling
  $\Longrightarrow$ no standard scaling, possibility for multi-scaling, multi-fractality, . . .
- equations for n-point functions in the large p regime
  $\Longrightarrow$ calculation of intermittency exponents

Conclusions

Summary

- FRG formalism to study turbulence from the NS equations
- exact relations between correlation functions from symmetries
- hints for the emergence of intermittency and multiscaling

Perpectives

- calculation of the deviations to Kolmogorov exponents
- study the inverse cascade of energy in $d = 2$
- ...

Thank you ! ! !