Critical behavior in spherical and hyperbolic spaces

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Motivations

- Quantum field theory on curved background
- Recent surge of interest on IR effects in dS spacetime [see Serreau's talk]
- Background dependence in FRG approach to asymptotically safe gravity:
  - $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and $\Delta_k S[\bar{g}; h] \Rightarrow \Gamma_k = \Gamma_k[\bar{g}; h]$, mWI
  - Nonperturbative (in $\bar{R}$) background effects in $f(R)$ approximation
- Usually we study asymptotic safety in Euclidean signature
  $\Rightarrow$ Euclidean QFT ($\equiv$ statistical field theory) on curved background
- In condensed matter the effect of curvature can be of interest for a number of reasons (e.g. for theoretical modeling of 3d frustration in simplified 2d models), and it has been studied in the context of liquids, percolation, Ising model, XY model, self-avoiding walks and more

This talk: How does curvature affect critical behavior in a simple model
and how do we see that with the FRGE

Outline

- Two simple backgrounds: d-dimensional spheres and hyperboloids
- Effective dimension and general expectations
- FRGE in the presence of background curvature

The d-dimensional sphere

- Homogeneous space: $S^d \simeq SO(d+1)/SO(d)$

$$\sum_{A=1}^{d+1} (X^A)^2 = a^2$$

$$ds^2_{(S^d)} = a^2 d\Omega_d = a^2 d\theta_d^2 + a^2 \sin^2(\theta_d) d\Omega_{d-1}$$

- Maximally symmetric:

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

with positive curvature:

$$\boxed{R = \frac{d(d-1)}{a^2}}$$

- Compact space $\Rightarrow$ discrete spectrum, including a zero mode

$$-\nabla^2 \psi_{n,j} = \frac{n(n+d-1)}{a^2} \psi_{n,j}$$

with multiplicity $D_n = \frac{(n+d-2)!\,(2n+d-1)}{n!(d-1)!}$, $j = 1, 2, ...D_n$, and $n = 0, 1, 2, ... + \infty$

The d-dimensional hyperboloid

- Homogeneous space: $H^d \simeq SO(d,1)/SO(d)$

$$\sum_{A=1}^{d}(X^A)^2 - (X^{d+1})^2 = -a^2$$

$$ds^2_{(H^d)} = d\tau^2 + a^2 \sinh^2(\tau/a) d\Omega_{d-1}$$

- Also maximally symmetric, but with negative curvature:

$$\boxed{R = -\frac{d(d-1)}{a^2}}$$

- Non-compact space $\Rightarrow$ continuous spectrum

$$-\nabla^2 \phi_{\lambda,l} = \frac{1}{a^2}(\lambda^2 + \rho^2)\varphi_{\lambda,l}$$

where $\rho = (d-1)/2$, $\lambda \in [0, +\infty)$, and $l = 0, 1, 2, ... + \infty$

note: no zero mode (not normalizable)

Effective dimension

Hausdorff dimension:

$$L^{d_H} = \int_L d^d x \sqrt{g}$$

where the integral extends over the set of points for which $\sigma(x, 0) \leq L$.

- Sphere: $d_H \to 0$ for $L \to \infty$ ($\sim$ it looks like a point from far)
- Hyperboloid: $d_H \to \infty$ for $L \to \infty$ (due to exponential growth e^L)

Spectral dimension $\Rightarrow$ same result

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Spectral dimension $\Rightarrow$ same result

$\Rightarrow$ We expect mean field behavior on hyperboloid, and no phase transition on sphere

(We reach same expectations by using Ginzburg criterion for scalar field)

FRGE on curved background – (I)

- LPA

$$\Gamma_k[\phi] = \int d^dx\sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V_k(\phi)\right]$$

- FRGE

$$k\partial_k V_k(\phi) = \frac{1}{2}\mathrm{Tr}_{(\mathcal{M})}\left[\frac{k\partial_k\mathcal{R}_k(-\nabla^2/k^2)}{-Z_k\nabla^2 + V_k''(\phi) + \mathcal{R}_k(-\nabla^2/k^2)}\right]_{|\phi=\text{const.}}$$

- Using the optimized cutoff and dimensionless variables

$$k\partial_k\tilde{V}_k(\tilde{\phi}) + d\,\tilde{V}_k(\tilde{\phi}) - \frac{d-2}{2}\tilde{\phi}\tilde{V}_k'(\tilde{\phi}) = \frac{1}{1+\tilde{V}_k''(\tilde{\phi})}F_{(\mathcal{M})}(\tilde{a})$$

$$\text{where}\quad \boxed{F_{(\mathcal{M})}(\tilde{a}) = \widetilde{\mathrm{Tr}}_{(\mathcal{M})}[\theta(1-\tilde{\Delta})]},\quad \text{and}\quad \tilde{a} = ak$$

$\Rightarrow$ All the background dependence is in the spectral counting function $F_{(\mathcal{M})}(\tilde{a})$.

FRGE on curved background – (II)

- In flat space, by Fourier transform:

$$F_{(E^3)}(\infty) = \frac{\Omega_{d-1}}{d\,(2\pi)^d} \xrightarrow[d=3]{} \frac{1}{6\pi^2}$$

- Hyperboloid ($d = 3$):

$$F_{(H^3)}(ak) = \frac{1}{6\pi^2}\left(1 - \frac{1}{a^2k^2}\right)^{\frac{3}{2}} \theta\left(1 - \frac{1}{a^2k^2}\right)$$

- Sphere ($d = 3$):

$$F_{(S^3)}(ak) = \frac{1}{2\pi^2 a^3 k^3}\mathcal{P}(\lfloor N_3 \rfloor)$$

where $\lfloor x \rfloor$ is the floor function,

$$\mathcal{P}(N) = \sum_{n=0}^{N} D_n = \frac{1}{6}(1+N)(2+N)(3+2N)$$

$$N_3 = -1 + \sqrt{1 + a^2k^2}$$

The spherical case gives rise to a staircase function, as a combined effect of the discrete spectrum and the use of a step function in the cutoff

Non-autonomous system

$$F_{(\mathcal{M})}(\tilde{a}) = \widetilde{\mathrm{Tr}}_{(\mathcal{M})}[\theta(1-\tilde{\Delta})]$$
$$\Downarrow$$

- Non-autonomous equation: explicit dependence on k via $\tilde{a} = ak$

  No rescaling of variables can turn the equation into an autonomous one

  $\Rightarrow$ Non-trivial fixed points are unlikely (k-dependence should factorize in β's)

- Non-autonomous equations found also in:

  quantum field theory at finite temperature [Tetradis, Wetterich - '93],
  non-commutative spacetime [Gurau, Rosten - '09],
  RG for matrix/tensor models [DB, BenGeloun, Oriti - to appear],
  gravity, if we eliminate G (treating it as inessential parameter) [Percacci, Perini - '04]

Scaling dimensions in the deep IR

Deep IR, $k \to 0$:

- Hyperboloid: $F_{(H^d)}(\tilde{a}) \to 0$ (due to mass gap)

  For $ak < (d-1)/2$ (due to optimized cutoff)

$$k\partial_k \tilde{V}_k(\tilde{\phi}) + d\,\tilde{V}_k(\tilde{\phi}) - \frac{d-2}{2}\tilde{\phi}\tilde{V}_k'(\tilde{\phi}) = 0$$

  $\equiv$ classical scaling equation

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- Sphere: $F_{(S^d)}(\tilde{a}) \to \infty$ (due to zero mode, and compactness)

  In order to absorb divergence of FRGE:

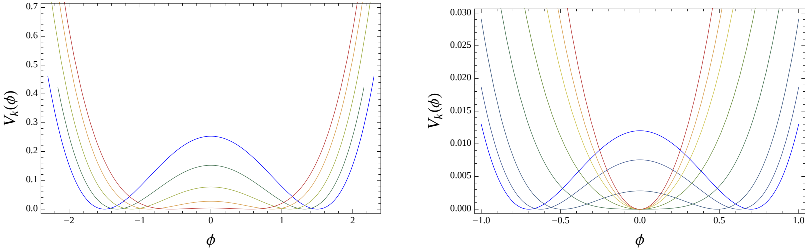
  $$\bar{\phi} = a^{d/2} k\,\phi\,, \quad \bar{V}(\bar{\phi}) = a^d V(a^{-d/2}k^{-1}\bar{\phi})$$

  The resulting equation for $k^2 < d/a^2$ is

  $$k\partial_k \bar{V}_k(\bar{\phi}) + \bar{\phi}\bar{V}_k'(\bar{\phi}) = \frac{1}{\Omega_d}\,\frac{1}{1+\bar{V}_k''(\bar{\phi})}$$

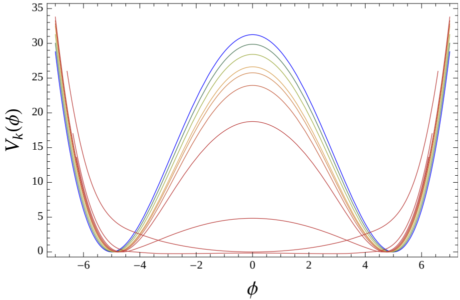
  $\equiv$ flat FRG equation for $d = 0$

Numerical integration – Flat space



- Solve numerically the flow equation, and integrating towards $k = 0$ observe different behavior as function of initial condition
- Blue curve: initial condition $V_\Lambda(\phi) = \lambda_\Lambda(\phi^2 - \rho_\Lambda)^2$

Numerical integration – Sphere

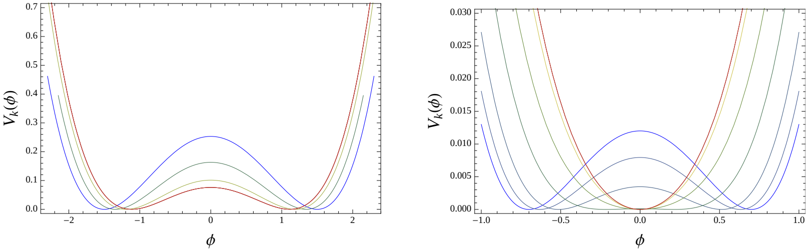


Despite the large value of the initial symmetry breaking parameter (here $\rho_\Lambda = 25$), symmetry restoration still takes place.

No true phase transition!

Numerical integration – Hyperboloid

Phase transition is there:



Note: no zero mode $\Rightarrow$ convexity of Γ does not imply convexity of the potential

Convexity of the effective action: all the eigenvalues of $\Gamma^{(2)}[\bar{\phi}]$ are non-negative

If $p^2 = 0$ is in the spectrum $\Rightarrow V''(\bar{\phi}) \geq 0$ (because $\Gamma^{(2)}[\bar{\phi}] = V''(\bar{\phi})$ at $p^2 = 0$)

In hyperbolic space the smallest eigenvalue of the Laplacian is $\nu_0 = \rho^2/a^2 > 0$ (with eigenfunction $\varphi_{0,l}$) $\Rightarrow \Gamma^{(2)}[\bar{\phi}] \cdot \varphi_{0,l} \neq V''(\bar{\phi}) \cdot \varphi_{0,l}$

In agreement with the mean field approximation, in which the potential in the broken phase is not convex

No nontrivial fixed points

A simple truncation

$$\tilde{V}_k(\tilde{\phi}) = v_0(k) + v_2(k)\,\tilde{\phi}^2 + v_4(k)\,\tilde{\phi}^4$$

$$\Downarrow$$

$$k\partial_k v_2 = -2\,v_2 - 12\,v_4\,\frac{F_{(\mathcal{M})}(\tilde{a})}{(1+2\,v_2)^2}$$

$$k\partial_k v_4 = (d-4)\,v_4 + 144\,v_4^2\,\frac{F_{(\mathcal{M})}(\tilde{a})}{(1+2\,v_2)^3}$$

$$\Downarrow$$

$$k\partial_k v_2^* = k\partial_k v_4^* = 0\ ? \quad \Rightarrow \quad v_2^* = \frac{4-d}{2d-32}\,, \quad v_4^* = \frac{12(d-4)}{(d-16)^3 F_{(\mathcal{M})}(\tilde{a})}$$

Hyperboloid: $v_4^* \to \infty$ for $k \to 0$
$\Rightarrow$ only Gaussian fixed point $\Rightarrow$ mean field exponents

Conclusions

- Strong IR effects produce very different physics on spheres and hyperboloids
- We have in these cases some general arguments (effective dimensionality, Ginzburg criterion) to give us some indications on what to expect
- FRGE can be used to nicely derive such properties
- Many possible calculations and extensions possible (η, large-N, other spaces...)
- Open question: can critical behavior be modified in a less trivial way by the background? (i.e. not explainable in terms of effective dimension)